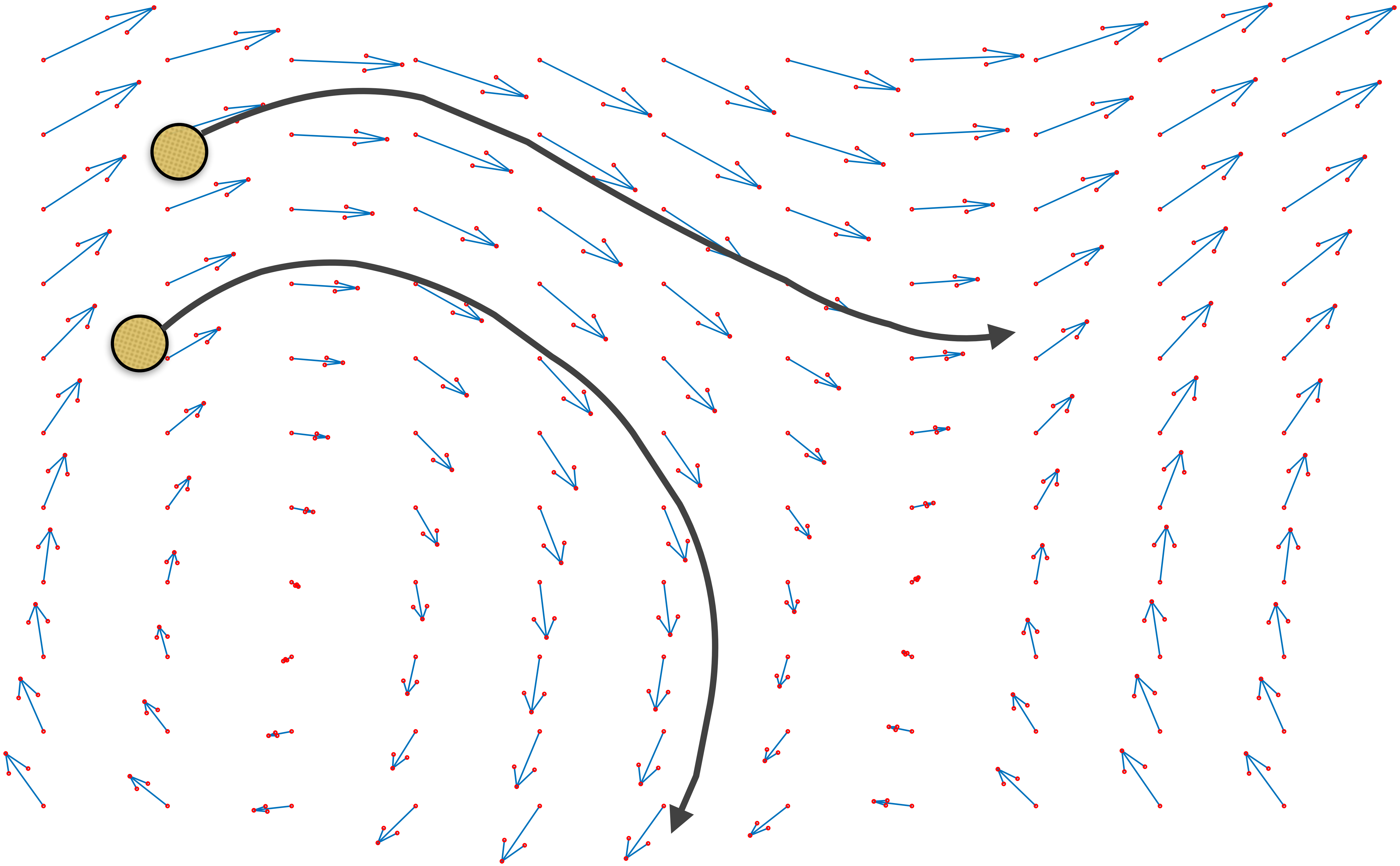
An Introduction to Physics-based Animation

Adam Bargteil
University of Maryland, Baltimore County

Tamar Shinar
University of California, Riverside

I. A Simple Start: Particle Dynamics

Let’s jump right in and consider the problem of tracing a particle through a velocity field



Initial Value Problem

$$\mathbf{x}_p(0) = \mathbf{x}_0$$

$$\frac{d\mathbf{x}_p(t)}{dt} = \mathbf{v}(\mathbf{x}_p, t)$$

Change ⟶ Difference

First-order Ordinary Differential Equation

Initial Value Problem

$$\mathbf{x}_p(0) = \mathbf{x}_0$$

$$\frac{d\mathbf{x}_p(t)}{dt} = \mathbf{v}(\mathbf{x}_p, t)$$

Simple

Powerful

Instructive

Euler’s Method

The Derivative

$$\frac{d\mathbf{x}_p(t)}{dt} = \lim_{\epsilon \to 0} \frac{\mathbf{x}_p(t+\epsilon) - \mathbf{x}_p(t)}{\epsilon}$$

$$\epsilon \longrightarrow \Delta t$$

$$\frac{d\mathbf{x}_p(t)}{dt} \approx \frac{\mathbf{x}_p(t+\Delta t) - \mathbf{x}_p(t)}{\Delta t}$$

Euler's Method

$$\frac{d\mathbf{x}_p(t)}{dt} \approx \frac{\mathbf{x}_p(t+\Delta t) - \mathbf{x}_p(t)}{\Delta t}$$

$$+$$

$$\frac{d\mathbf{x}_p(t)}{dt} = \mathbf{v}(\mathbf{x}_p, t)$$

$$\downarrow$$

$$\frac{\mathbf{x}_p(t+\Delta t) - \mathbf{x}_p(t)}{\Delta t} = \mathbf{v}(\mathbf{x}_p, t)$$

$$\downarrow$$

$$\mathbf{x}_p(t+\Delta t) = \mathbf{x}_p(t) + \Delta t \cdot \mathbf{v}(\mathbf{x}_p, t)$$

The Great Tradeoff

$$\frac{d\mathbf{x}_p(t)}{dt} \approx \frac{\mathbf{x}_p(t+\Delta t) - \mathbf{x}_p(t)}{\Delta t}$$

As Δt decreases

the approximation gets better

but

the computational cost increases

Let’s consider another problem

In the real world

$$\mathbf{f} = m\mathbf{a}$$

Another Initial Value Problem

$$\mathbf{x}_p(0) = \mathbf{x}_0$$

$$\frac{d^2\mathbf{x}_p(t)}{dt^2} = \frac{\mathbf{f}(\mathbf{x}_p, t)}{m_p}$$

Second-order Ordinary Differential Equation

Another Initial Value Problem

$$\mathbf{x}_p(0) = \mathbf{x}_0$$

$$\mathbf{v}_p(0) = \mathbf{v}_0$$

$$\frac{d\mathbf{x}_p(t)}{dt} = \mathbf{v}(\mathbf{x}_p, t)$$

$$\frac{d\mathbf{v}_p(t)}{dt} = \frac{\mathbf{f}(\mathbf{x}_p, t)}{m_p}$$

Coupled First-order Ordinary Differential Equations

Euler's Method (Again)

$$\mathbf{v}_p(t + \Delta t) = \mathbf{v}_p(t) + \Delta t \cdot \frac{\mathbf{f}(\mathbf{x}_p, t)}{m_p}$$

$$\mathbf{x}_p(t + \Delta t) = \mathbf{x}_p(t) + \Delta t \cdot \mathbf{v}_p(t + \Delta t)$$

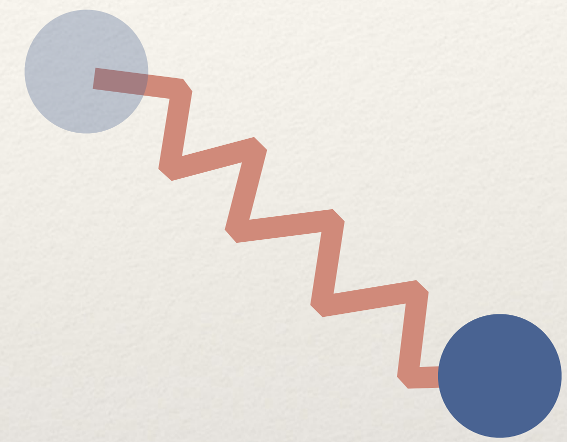
Symplectic Euler

```
struct Particle {
  double mass;
  Eigen::Vector3d pos, vel, frc;
};


foreach (p : particles) {
  p.frc = 0.0;
}

foreach (f : forces) {
  foreach (p : forces.affectedParticles) {
    p.frc += f.computeForce(p);
  }
}

foreach (p : particle) {
  p.vel += dt * p.frc / p.mass;
  p.pos += dt * p.vel;
}
```

Check our Karl Sim’s *Particle Dreams*

Let's Add Springs!

Springs



$$\mathbf{f} = -k\mathbf{d}$$

$$\mathbf{f}_p = -k\mathbf{x}_p$$

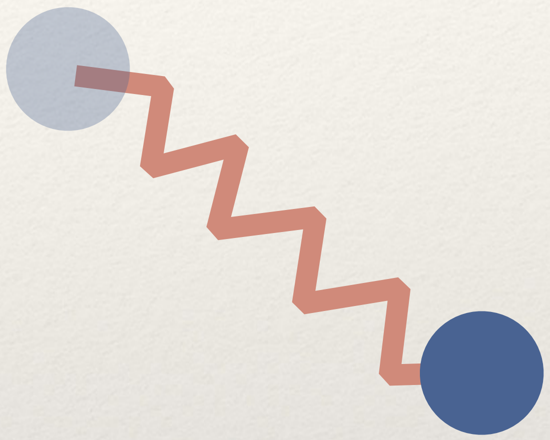
non-zero rest length?

$$\mathbf{f}_p = -k\left(\|\mathbf{x}_p\|\right)\frac{\mathbf{x}_p}{\|\mathbf{x}_p\|}$$

$$\mathbf{f}_p = -k\left(\|\mathbf{x}_p\| - r\right)\frac{\mathbf{x}_p}{\|\mathbf{x}_p\|}$$

Springs

$$\mathbf{f}_p = -k\left(\|\mathbf{x}_p\| - r\right)\frac{\mathbf{x}_p}{\|\mathbf{x}_p\|}$$

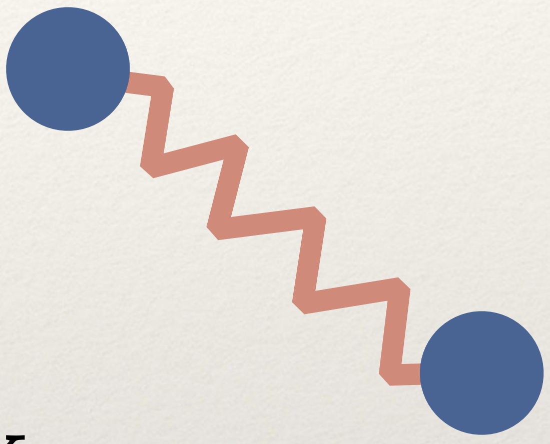


Strain $\left(\frac{\|\mathbf{x}_p\|}{r} - 1\right)$

$$\mathbf{f}_p = -k\left(\frac{\|\mathbf{x}_p\|}{r} - 1\right)\frac{\mathbf{x}_p}{\|\mathbf{x}_p\|}$$

Springs

$$\mathbf{f}_p = -k\left(\frac{\|\mathbf{x}_p\|}{r} - 1\right)\frac{\mathbf{x}_p}{\|\mathbf{x}_p\|}$$



$$\mathbf{f}_p = k\left(\frac{\|\mathbf{x}_q - \mathbf{x}_p\|}{r} - 1\right)\frac{\mathbf{x}_q - \mathbf{x}_p}{\|\mathbf{x}_q - \mathbf{x}_p\|}$$

arbitrary connection?

$$\mathbf{f}_q = -\mathbf{f}_p$$

Damping

$$\mathbf{f}_p = k \left(\frac{\|\mathbf{x}_q - \mathbf{x}_p\|}{r} - 1 \right) \frac{\mathbf{x}_q - \mathbf{x}_p}{\|\mathbf{x}_q - \mathbf{x}_p\|}$$

$$\mathbf{f}_p = k_d \left(\underbrace{\frac{\mathbf{v}_q - \mathbf{v}_p}{r}}_{\text{relative velocity}} \cdot \underbrace{\frac{\mathbf{x}_q - \mathbf{x}_p}{\|\mathbf{x}_q - \mathbf{x}_p\|}}_{\text{spring direction}} \right) \frac{\mathbf{x}_q - \mathbf{x}_p}{\|\mathbf{x}_q - \mathbf{x}_p\|}$$

$$\mathbf{f}_p = \left[k_s \left(\frac{\|\mathbf{x}_q - \mathbf{x}_p\|}{r} - 1 \right) + k_d \left(\frac{(\mathbf{v}_q - \mathbf{v}_p) \cdot (\mathbf{x}_q - \mathbf{x}_p)}{r\|\mathbf{x}_q - \mathbf{x}_p\|} \right) \right] \frac{\mathbf{x}_q - \mathbf{x}_p}{\|\mathbf{x}_q - \mathbf{x}_p\|}$$

```
foreach (p : particles) {
  p.frc = 0.0;
}

foreach (s : springs) {
  Eigen::Vector3d d = particles[s->j].pos - particles[s->i].pos;
  double l = d.norm();
  Eigen::Vector3d v = particles[s->j].vel - particles[s->i].vel;
  Eigen::Vector3d frc = (params.k_s*((l / s->r) - 1.0) +
     params.k_d*(v.dot(d)/(l*s->r))) * (d/l);
  particles[s.i].frc += frc
  particles[s.j].frc -= frc
}

foreach (p : particle) {
  p.vel += dt * p.frc / p.mass;
  p.pos += dt * p.vel;
}
```

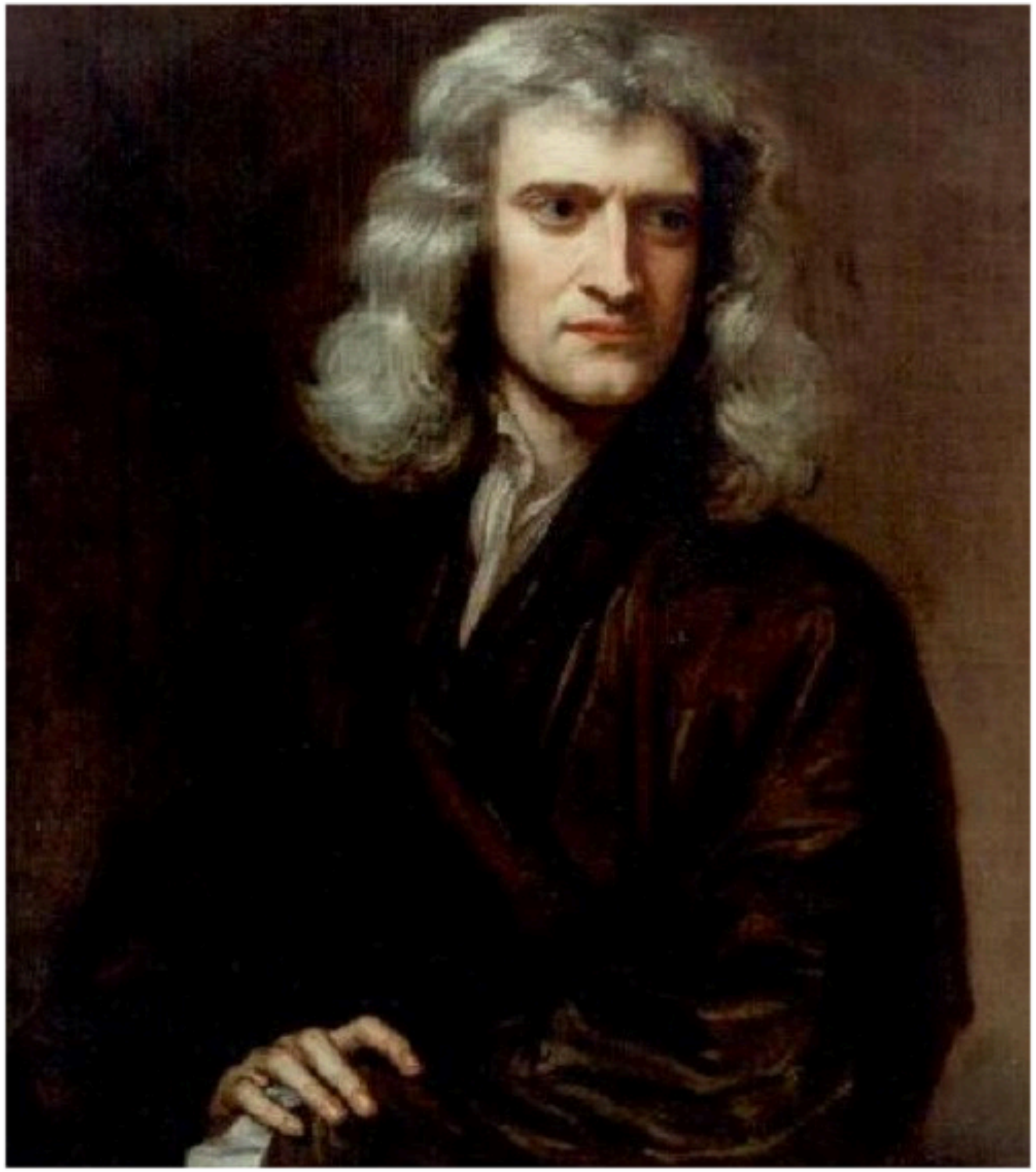
Live Demo

II. Mathematical Models

Newton's Laws

Newtonian Mechanics

- Published in *Principia,* 1687
- Includes three laws of motion:
  - inertia
  - f = ma
  - action/reaction
- Idealized particle or point mass



Newton's First Law

A body persists at rest or in uniform motion in a straight line unless acted upon by a force

- Law of Inertia
  - Defines an inertial frame of reference

Newton's Second Law (f = ma)

The rate of change of momentum of a body is directly proportional to the force applied to the body

Newton's Second Law (f = ma)

The rate of change of momentum of a body is directly proportional to the force applied to the body

$$\mathbf{p}(t)$$

Newton's Second Law (f = ma)

The rate of change of momentum of a body is directly proportional to the force applied to the body

$$m\mathbf{v}(t)$$

Newton's Second Law (f = ma)

The rate of change of momentum of a body is directly proportional to the force applied to the body

$$\frac{d}{dt} m\mathbf{v}(t)$$

Newton’s Second Law (f = ma)

The rate of change of momentum of a body is directly proportional to the force applied to the body

$$\frac{d}{dt} m\mathbf{v}(t) = m \frac{d}{dt} \mathbf{v}(t)$$

Newton's Second Law (f = ma)

The rate of change of momentum of a body is directly proportional to the force applied to the body

$$\frac{d}{dt} m\mathbf{v}(t) = m\mathbf{a}(t)$$

Newton's Second Law (f = ma)

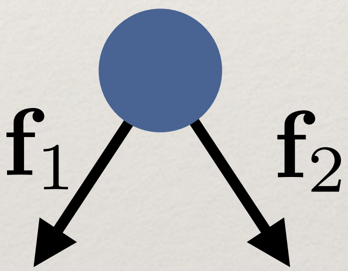
The rate of change of momentum of a body is directly proportional to the force applied to the body

$$\frac{d}{dt} m\mathbf{v}(t) = m\mathbf{a}(t) = \mathbf{f}(t)$$

Newton's Second Law (f = ma)

The rate of change of momentum of a body is directly proportional to the force applied to the body

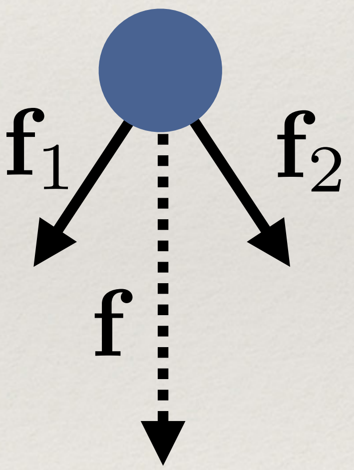
$$\frac{d}{dt} m\mathbf{v}(t) = m\mathbf{a}(t) = \mathbf{f}(t)$$



Newton's Second Law (f = ma)

The rate of change of momentum of a body is directly proportional to the force applied to the body

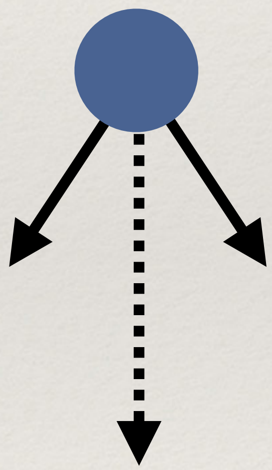
$$\frac{d}{dt} m\mathbf{v}(t) = m\mathbf{a}(t) = \mathbf{f}(t)$$



Newton's Second Law (f = ma)

The rate of change of momentum of a body is directly proportional to the force applied to the body

$$\frac{d}{dt}m\mathbf{v}(t) = m\mathbf{a}(t) = \mathbf{f}(t)$$



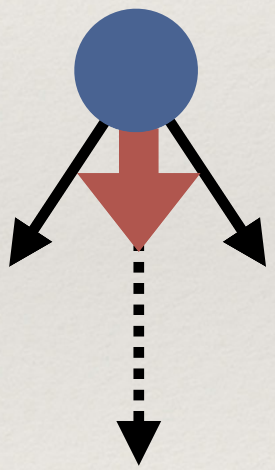
- acceleration is inversely proportional to mass

$$\mathbf{a}(t) = \frac{1}{m}\mathbf{f}(t)$$

Newton's Second Law (f = ma)

The rate of change of momentum of a body is directly proportional to the force applied to the body

$$\frac{d}{dt} m\mathbf{v}(t) = m\mathbf{a}(t) = \mathbf{f}(t)$$



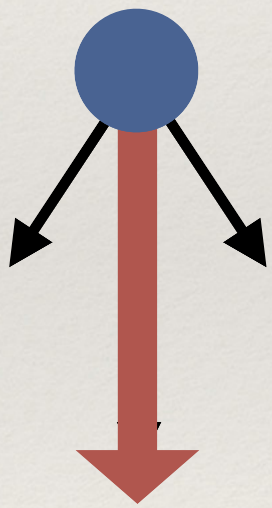
- acceleration is inversely proportional to mass

$$\mathbf{a}(t) = \frac{1}{m}\mathbf{f}(t)$$

Newton's Second Law (f = ma)

The rate of change of momentum of a body is directly proportional to the force applied to the body

$$\frac{d}{dt}m\mathbf{v}(t) = m\mathbf{a}(t) = \mathbf{f}(t)$$



- acceleration is inversely proportional to mass

$$\mathbf{a}(t) = \frac{1}{m}\mathbf{f}(t)$$

Newton's Second Law (f = ma)

The rate of change of momentum of a body is directly proportional to the force applied to the body

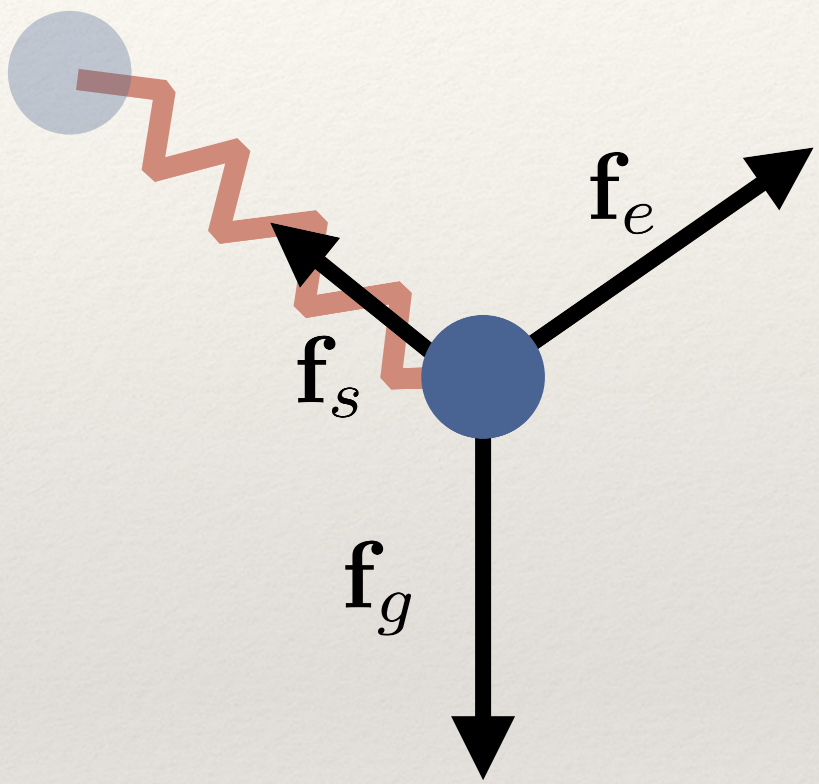
$$\frac{d}{dt}m\mathbf{v}(t) = m\mathbf{a}(t) = \mathbf{f}(t)$$

- An **evolution** equation for a system of interacting particles

Newton's Second Law (f = ma)

- Second order ODE

$$m\ddot{\mathbf{x}}(t) = \mathbf{f}(t)$$

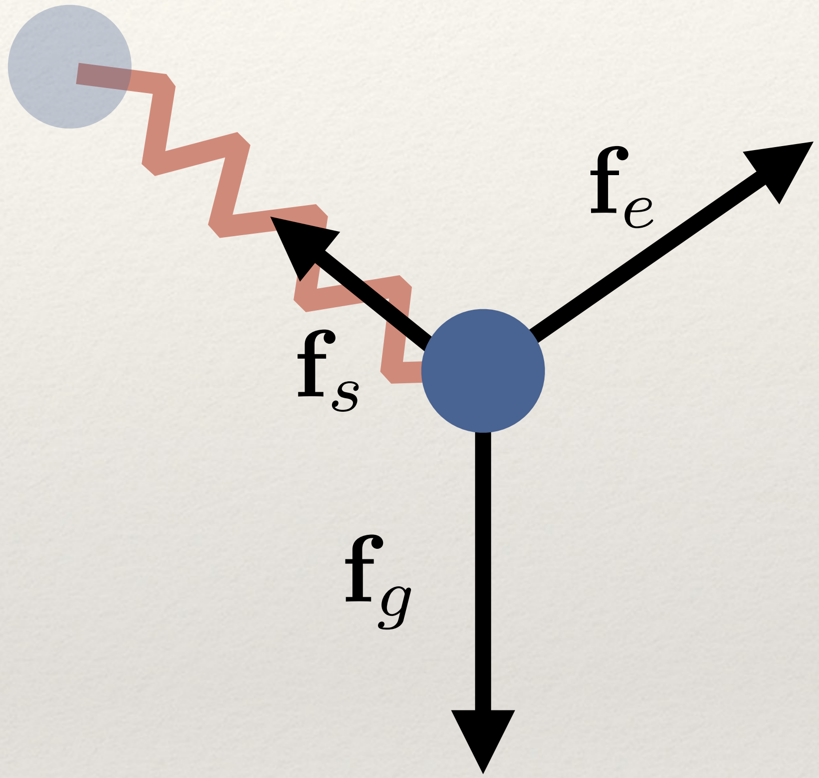


Newton's Second Law (f = ma)

- First order system of ODEs

$$\dot{\mathbf{x}}(t) = \mathbf{v}(t)$$

$$\dot{\mathbf{v}}(t) = \mathbf{a}(t) = \frac{1}{m}\mathbf{f}(t)$$



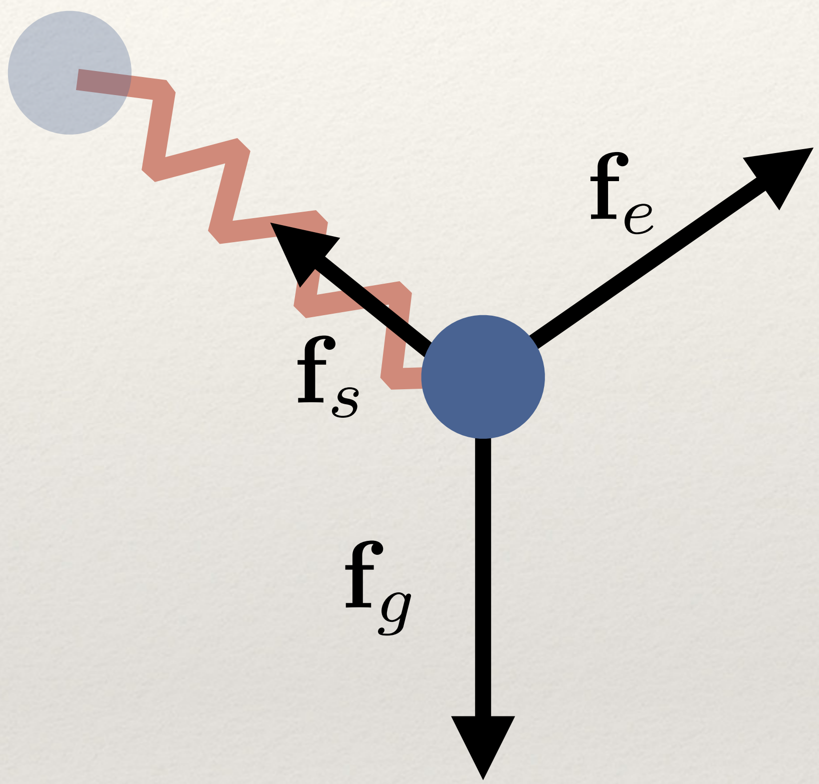
Newton's Second Law (f = ma)

- First order system of ODEs

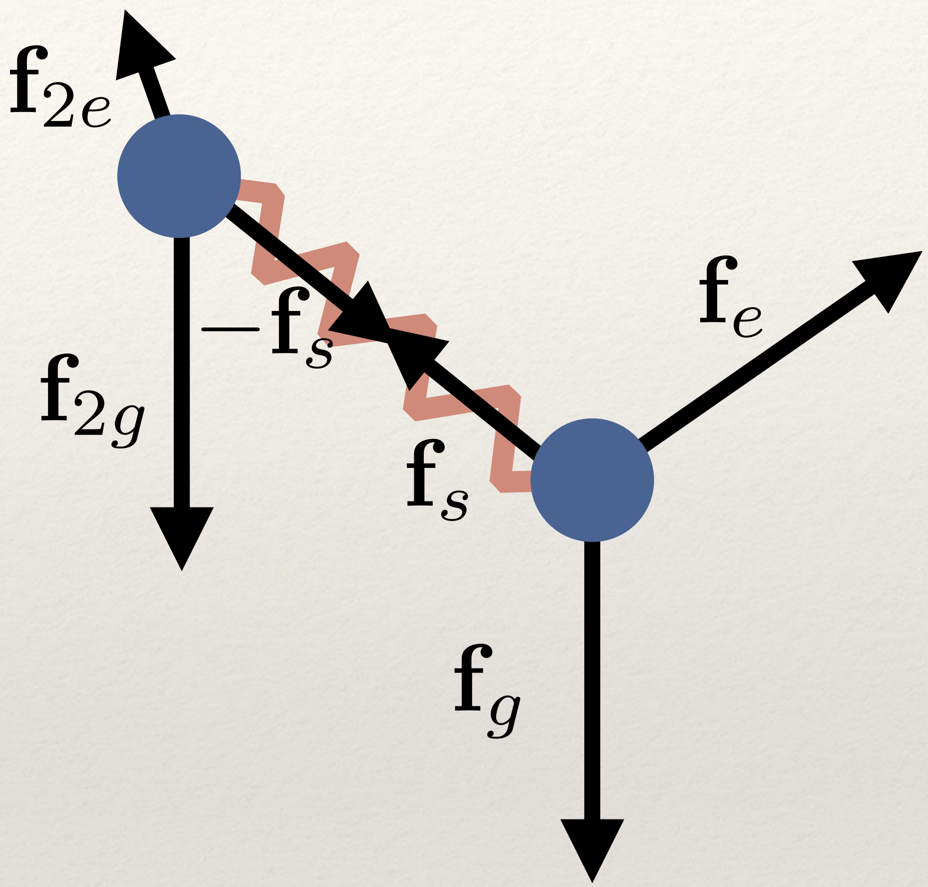
$$\dot{\mathbf{x}}(t) = \mathbf{v}(t)$$

$$\dot{\mathbf{v}}(t) = \mathbf{a}(t) = \frac{1}{m}\mathbf{f}(t)$$

$$\mathbf{f}(t) = \mathbf{f}_e(t) + \mathbf{f}_g(t) + \mathbf{f}_s(t)$$



Newton's Second Law (f = ma)



- To model a system of particles,
  - characterize all the forces on each particle
  - Start with some initial conditions and apply f = ma to evolve in time

Newton's Third Law (Action/Reaction)

For every action, there is an equal and opposite reaction

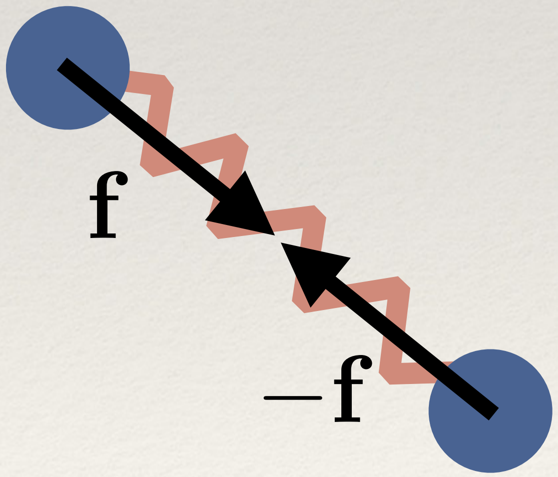
- If body 1 applies force **f** to body 2, then body 2 applies force **-f** to body 1

Newton's Third Law (Action/Reaction)

For every action, there is an equal and opposite reaction

- If body 1 applies force **f** to body 2, then body 2 applies force **-f** to body 1

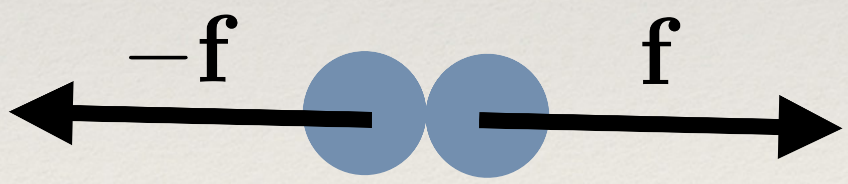
- Example: Two particles connected by a spring force, equal/opposite pair of forces



Newton's Third Law (Action/Reaction)

For every action, there is an equal and opposite reaction

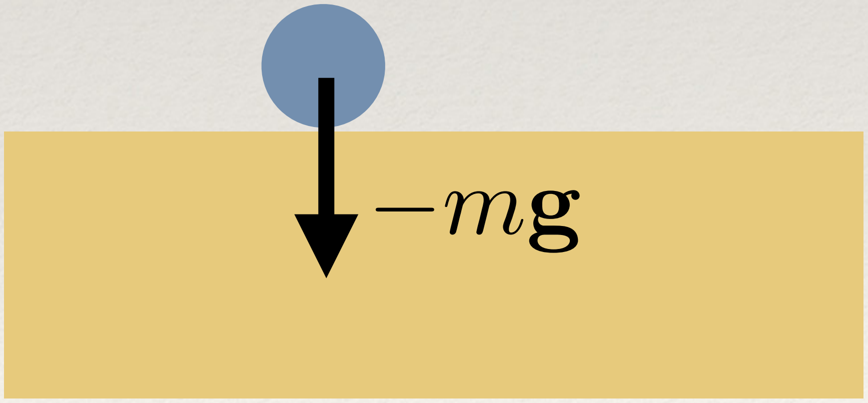
- If body 1 applies force **f** to body 2, then body 2 applies force **-f** to body 1
- Example: In the collision of two particles, equal/opposite pair of forces prevents interpenetration



Newton's Third Law (Action/Reaction)

For every action, there is an equal and opposite reaction

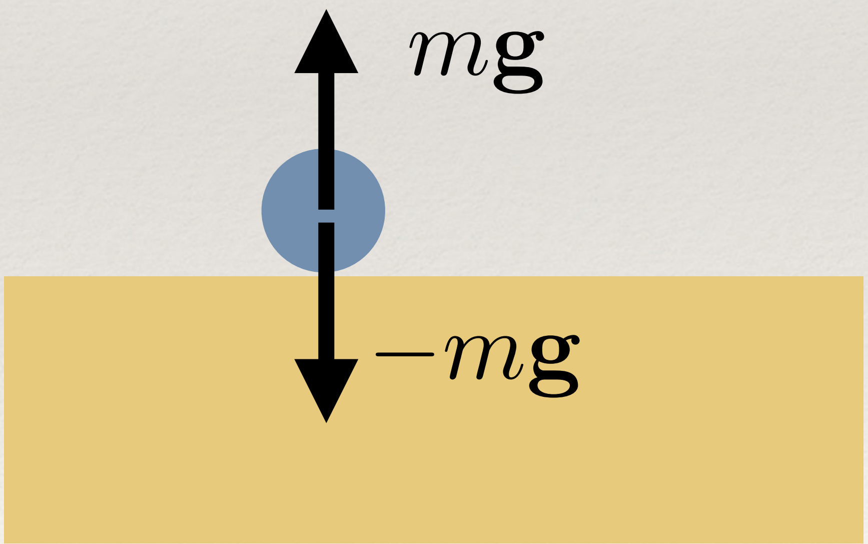
- If body 1 applies force **f** to body 2, then body 2 applies force **-f** to body 1
- Example: Particle resting on a surface, equal/opposite pair of forces prevents interpenetration



Newton's Third Law (Action/Reaction)

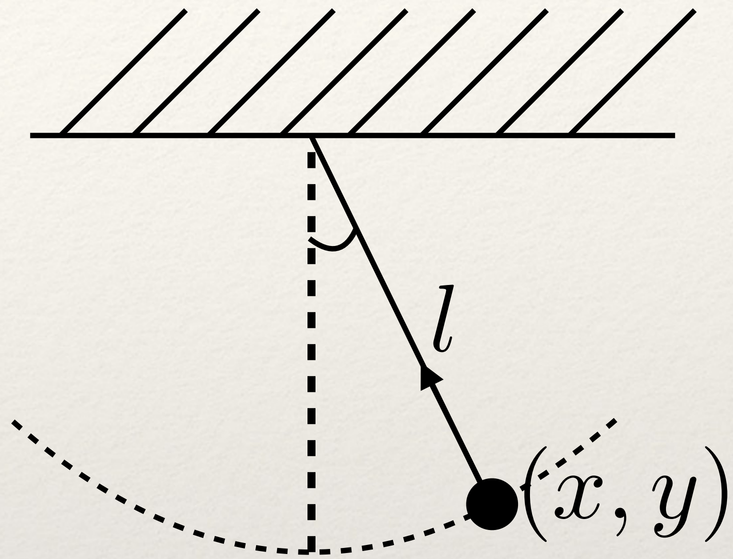
For every action, there is an equal and opposite reaction

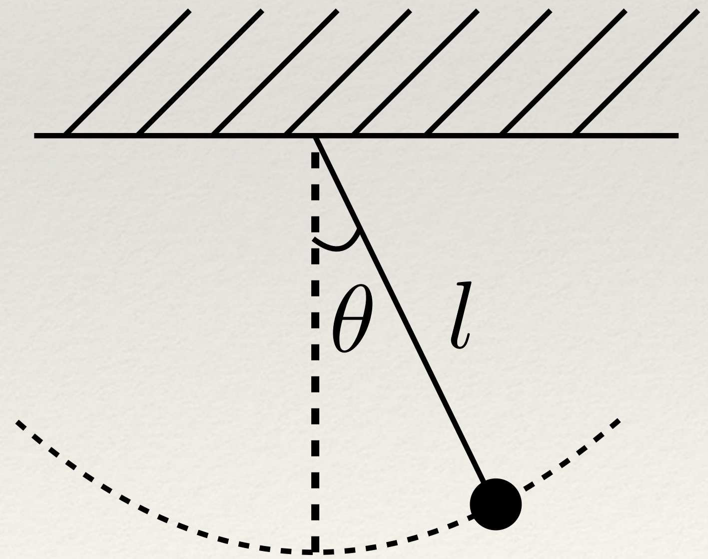
- If body 1 applies force **f** to body 2, then body 2 applies force **-f** to body 1
- Example: Particle resting on a surface, equal/opposite pair of forces prevents interpenetration



Alternative: Analytical Mechanics

- **Newtonian Mechanics** is one formulation of classical mechanics
  - Based on vectors in Cartesian space
- Another set of approaches is called **Analytical Mechanics** and is based on a principle of least action
  - variational approaches let you use any set of coordinates





Conservation Laws

Conserved Quantities

- Cannot be created or destroyed!
- Includes mass, linear momentum, angular momentum, and energy

Conservation Laws

- Used in deriving evolution equations
- Inform choice of discrete approximation to continuous equations
- Implications for visual quality, numerical accuracy, and stability

Conservation of Mass

- Mass not created or destroyed (inexact)
- Mass naturally conserved in particle-based methods
  - Particles carry mass with them as they move
- Grid-based methods sometimes have issues with proper mass conservation

Conservation of Momentum

- By Newton's second law, if there is no net force on a body, i.e., $\mathbf{f} = \mathbf{0}$

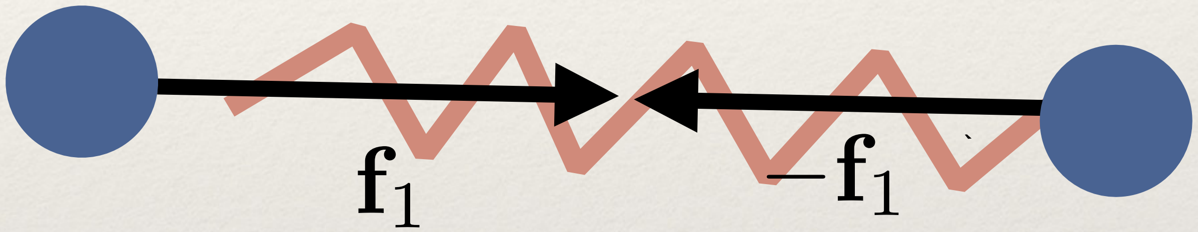
$$\frac{d}{dt} m\mathbf{v} = \mathbf{0}$$

$$\Rightarrow m\mathbf{v}(t) = \text{constant}$$

- So the momentum of the particle is conserved
- Similarly, if there is no net torque on a body, angular momentum is constant

Conservation of Momentum

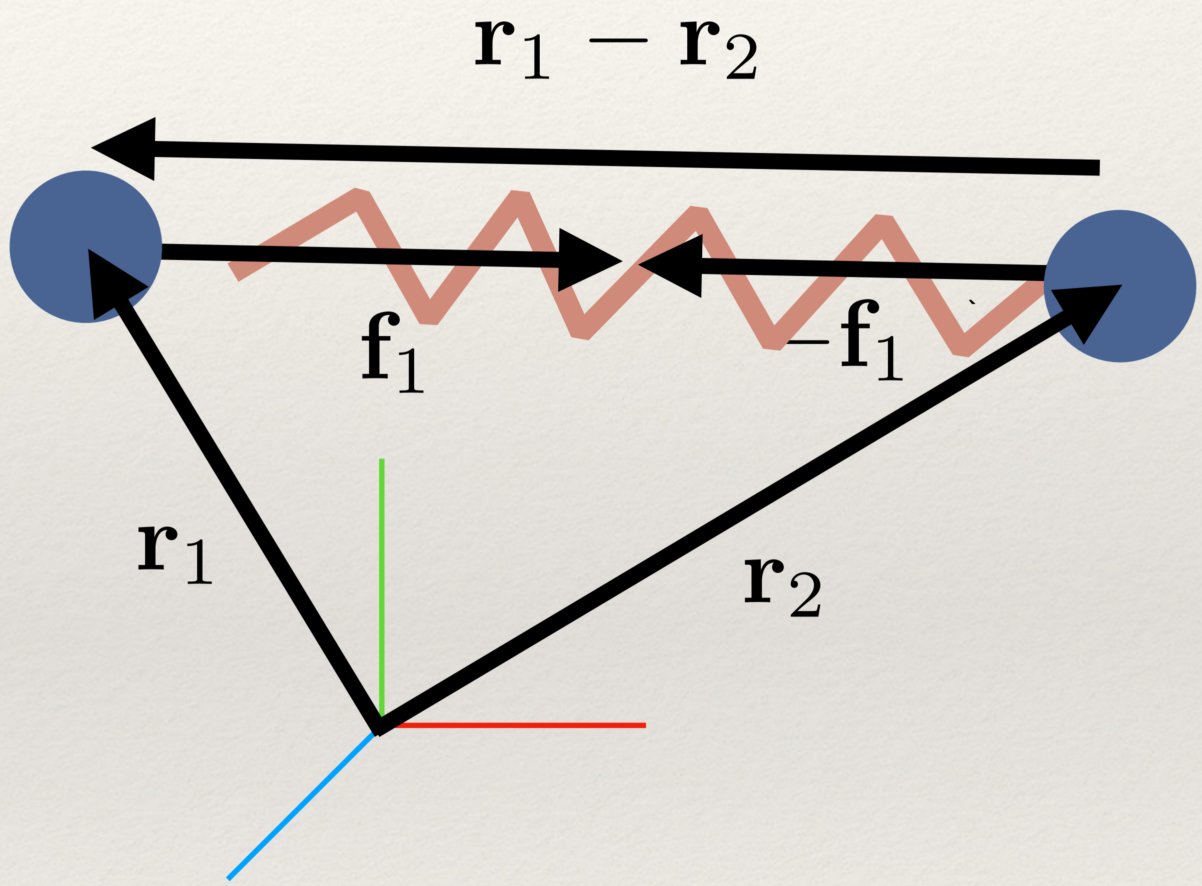
- Newton's third law equal/opposite also implies conservation of linear and angular momentum



$$\frac{d}{dt}\mathbf{P}(t) = \mathbf{f}_1 + (-\mathbf{f}_1) = \mathbf{0}$$

Conservation of Momentum

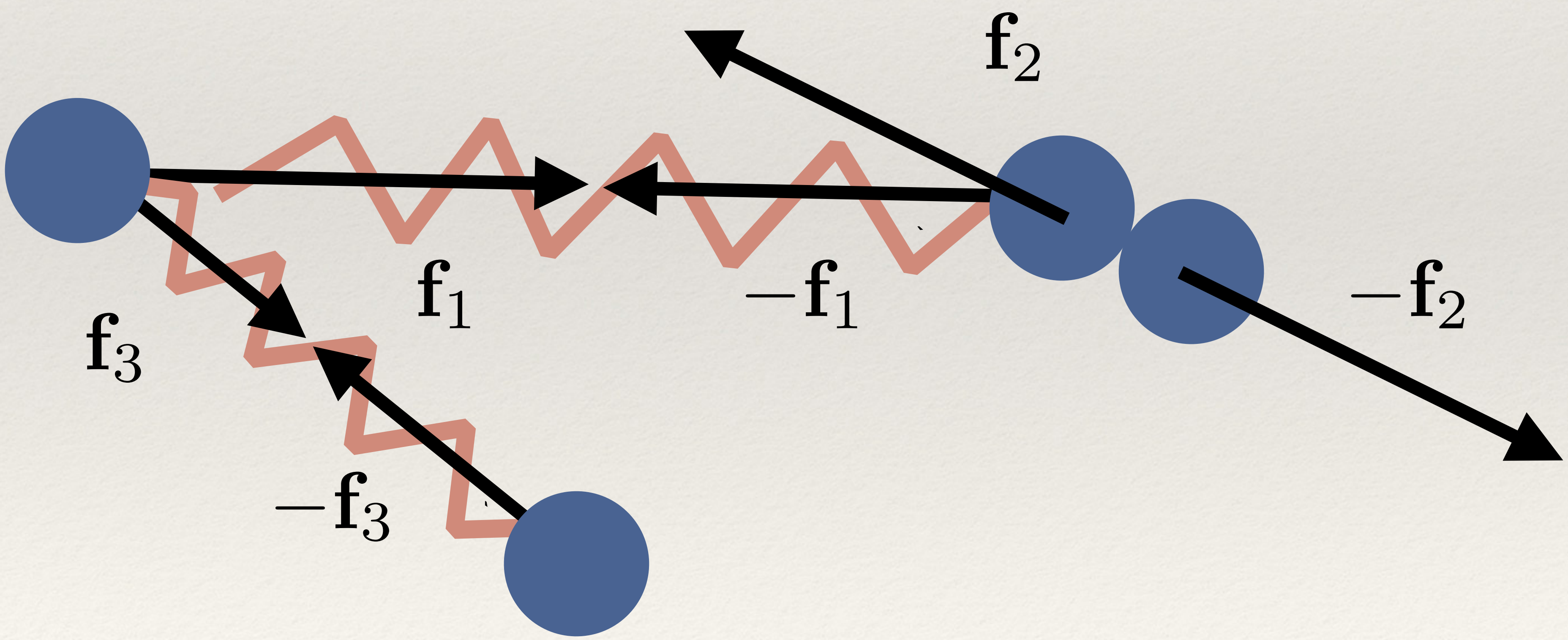
- Newton's third law equal/opposite also implies conservation of linear and angular momentum



$$\frac{d}{dt}\mathbf{L}(t) = \mathbf{r}_1 \times \mathbf{f}_1 + \mathbf{r}_2 \times (-\mathbf{f}_1) = (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{f}_1 = \mathbf{0}$$

Conservation of Momentum

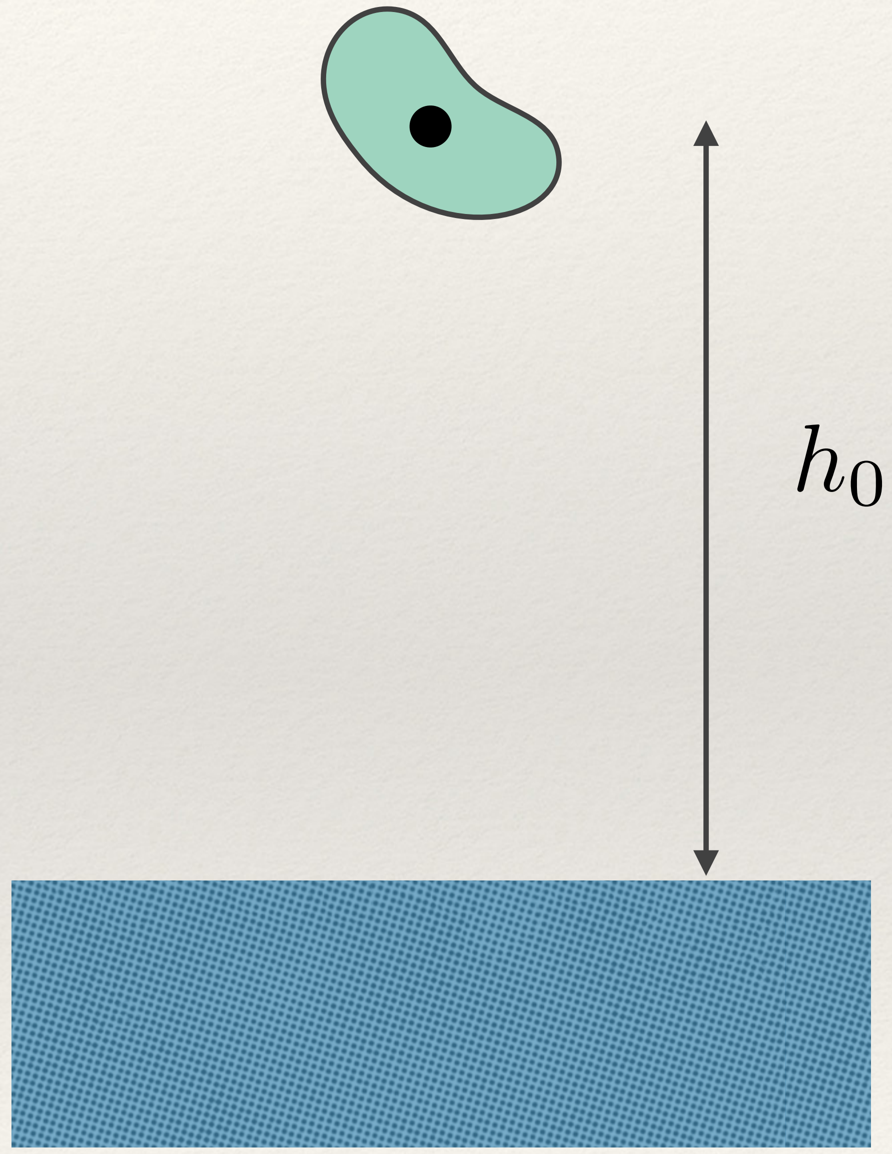
- Same holds for a collection of interacting particles!



Conservation of Energy

- Initial energy (potential)

$$mgh_0$$



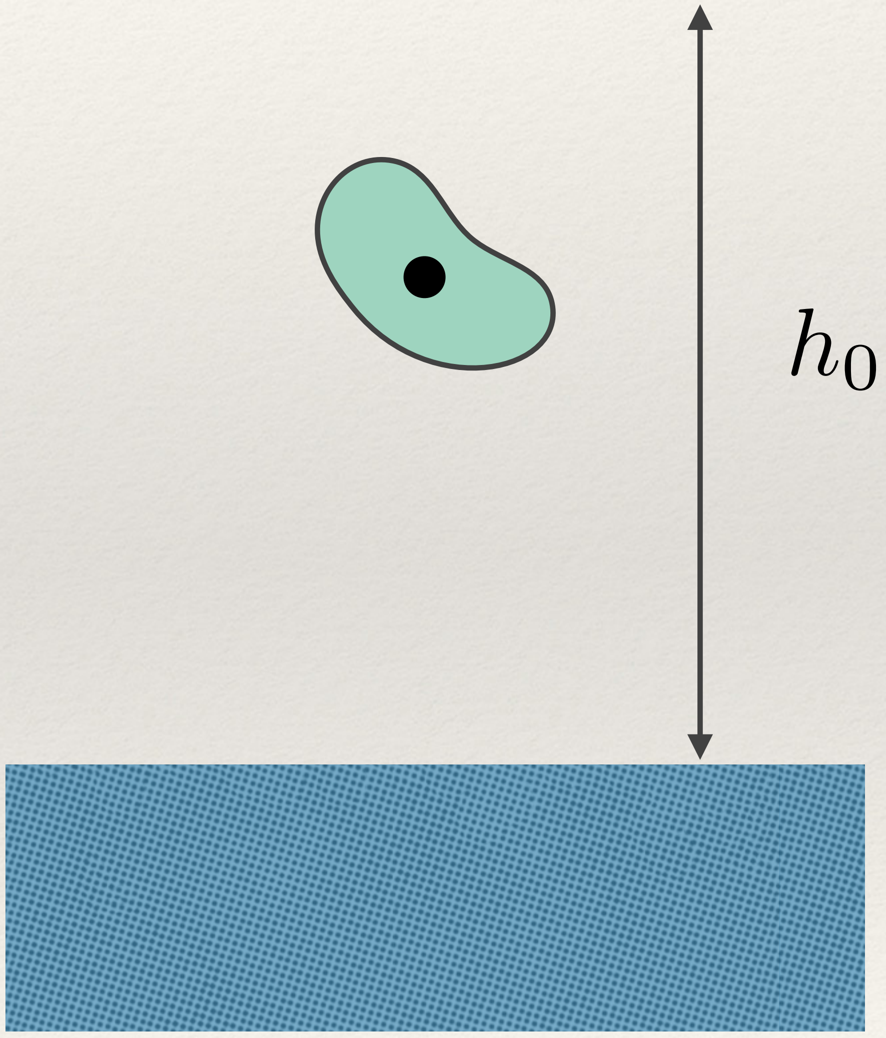
Conservation of Energy

- Initial energy (potential)

$$mgh_0$$

- Conservation of Energy

$$\frac{1}{2}mv(t)^2 + mgh(t) = mgh_0$$



Conservation of Energy

- Initial energy (potential)

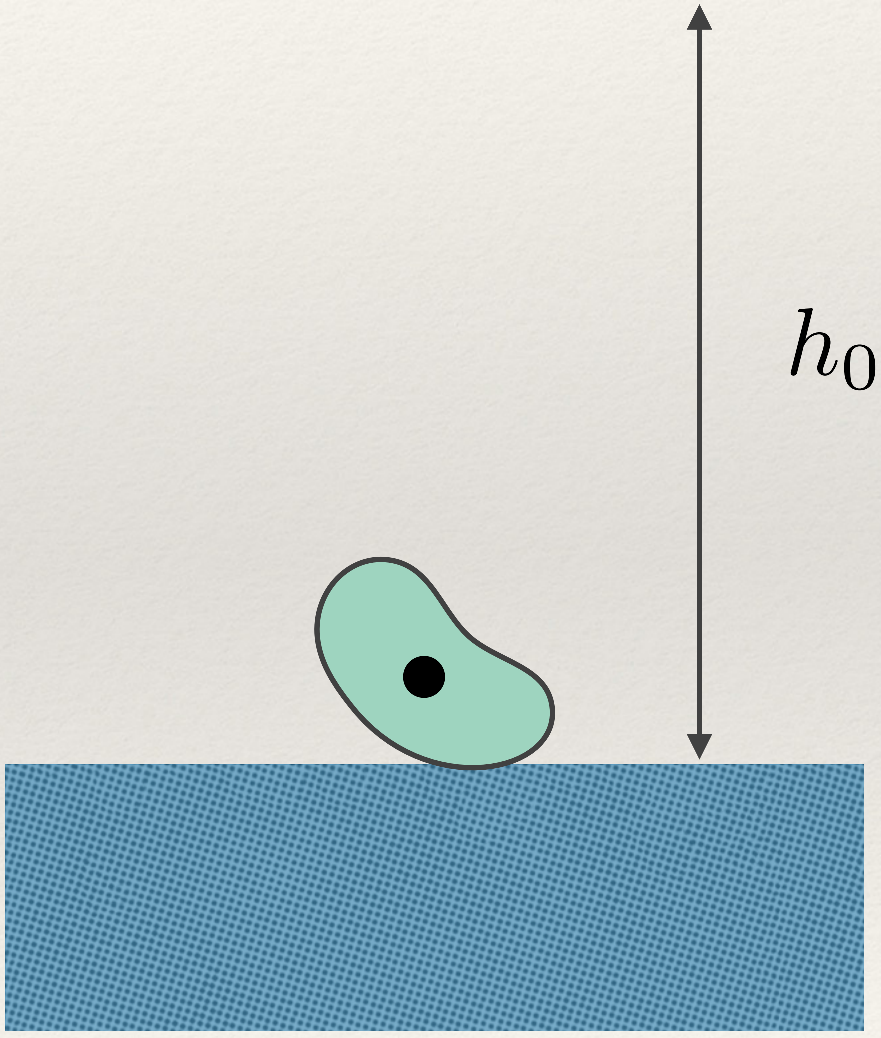
$$mgh_0$$

- Conservation of Energy

$$\frac{1}{2}mv(t)^2 + mgh(t) = mgh_0$$

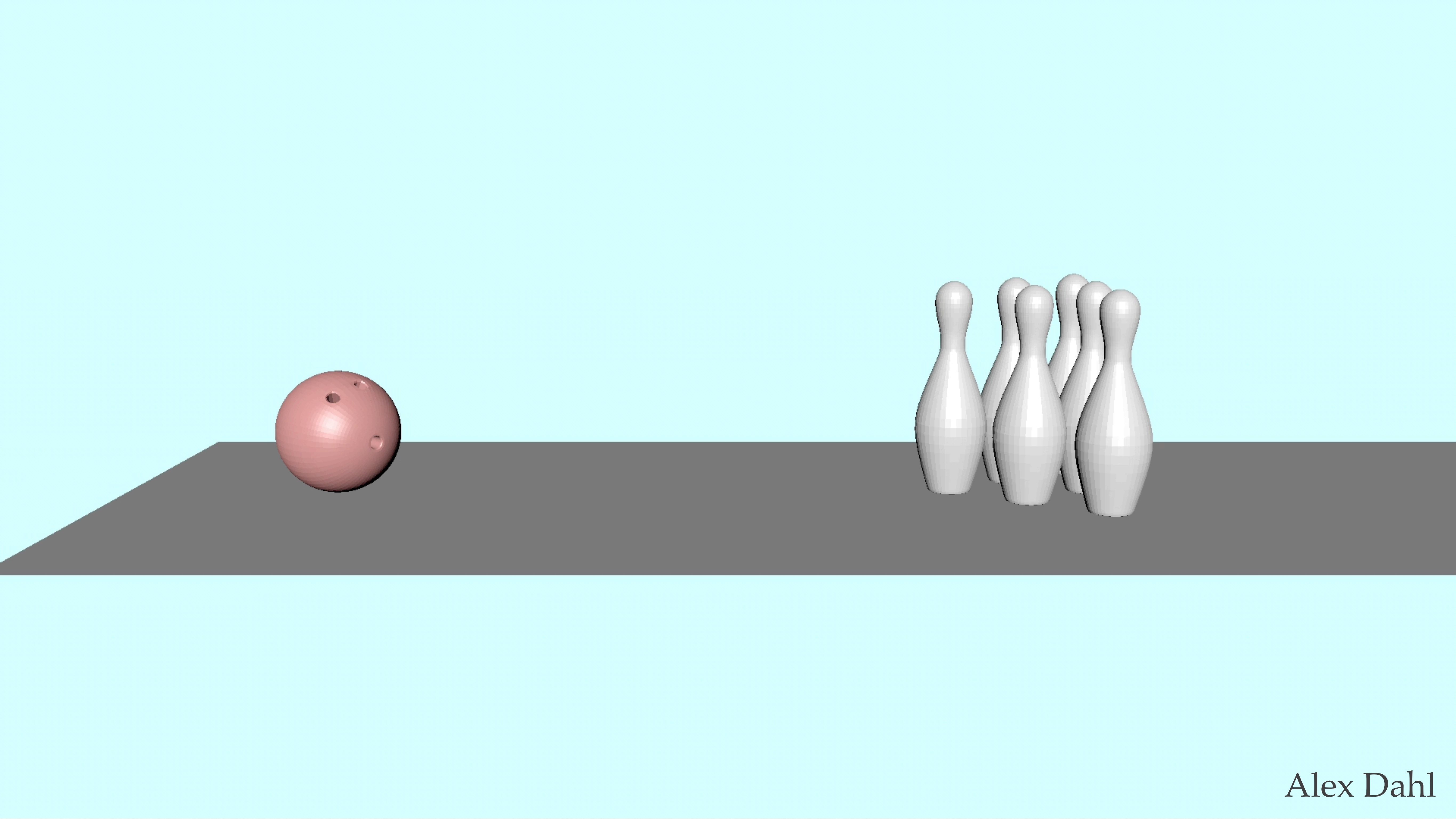
- Speed when hits

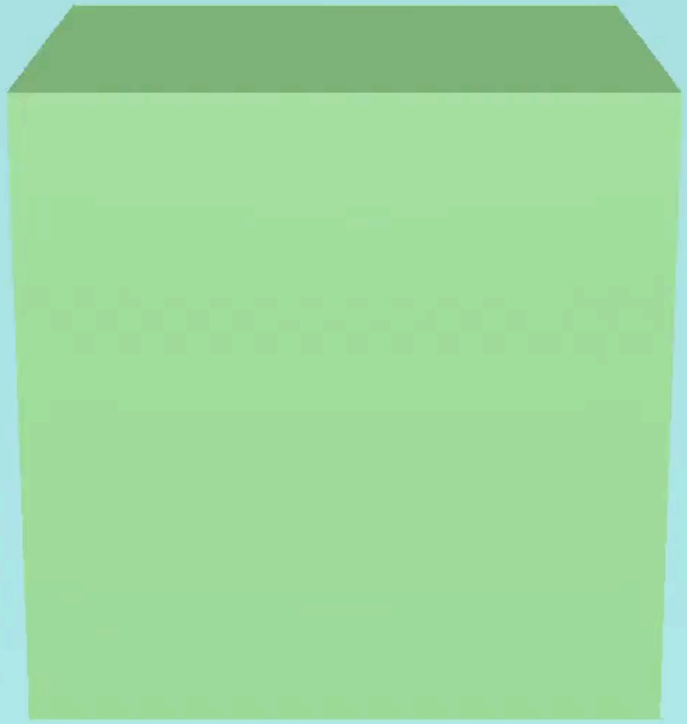
$$v(t) = \sqrt{2gh_0}$$



Numerical Conservation

- Different numerical schemes have different momentum and energy conservation properties
- In many schemes, momentum and/or energy may grow or decay nonphysically
  - instability (blow up), or
  - motion too damped

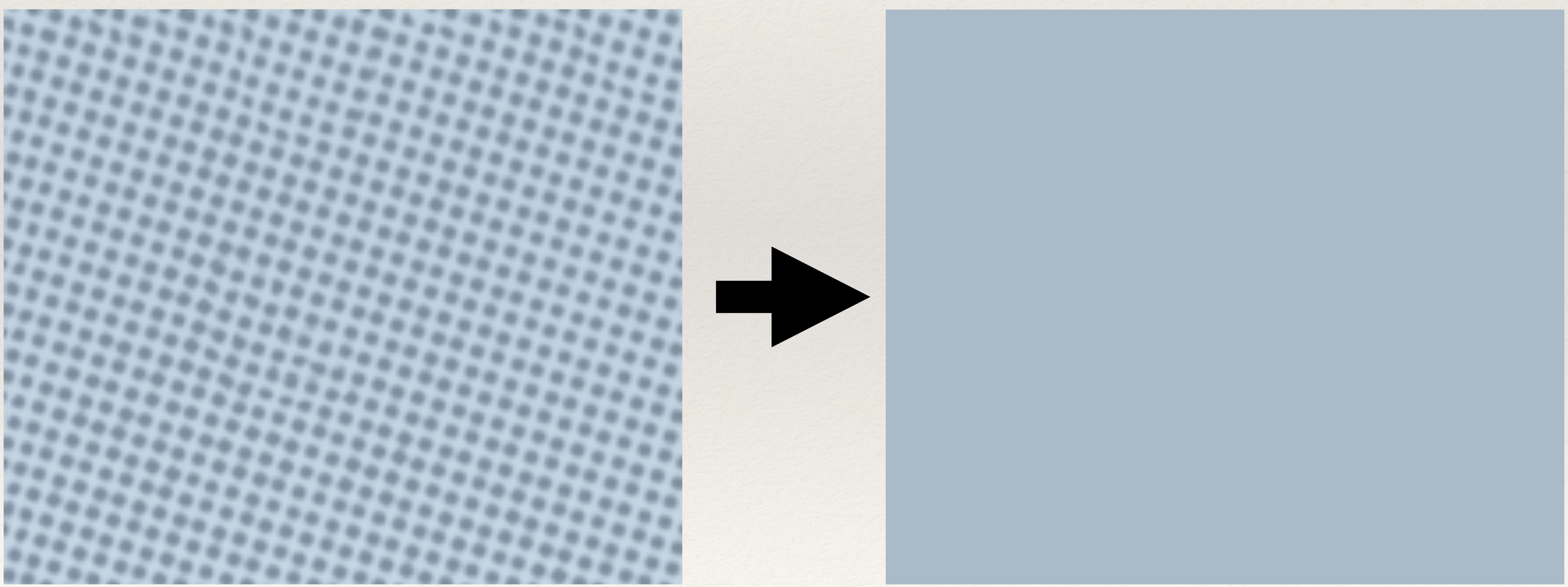

Alex Dahl



Alex Dahl

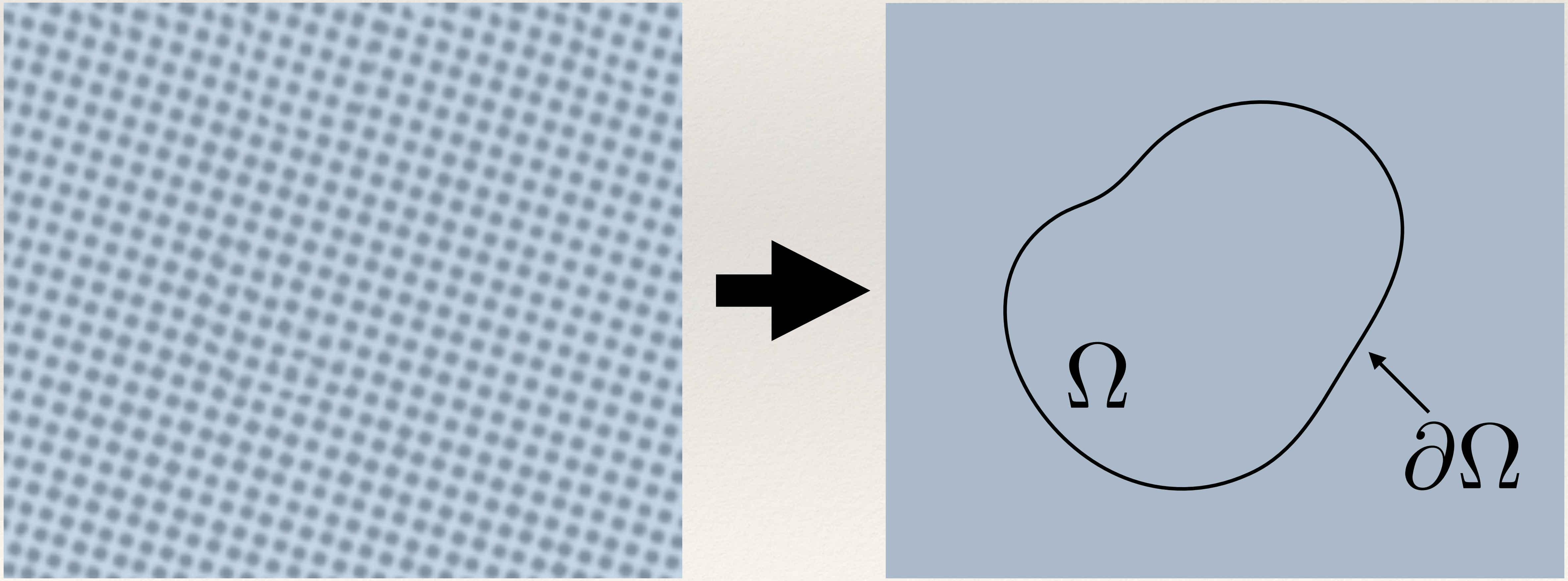
Conservation Laws for Continua

- At large enough length scales, model matter as a continuum rather than set of discrete particles



Conservation Laws for Continua

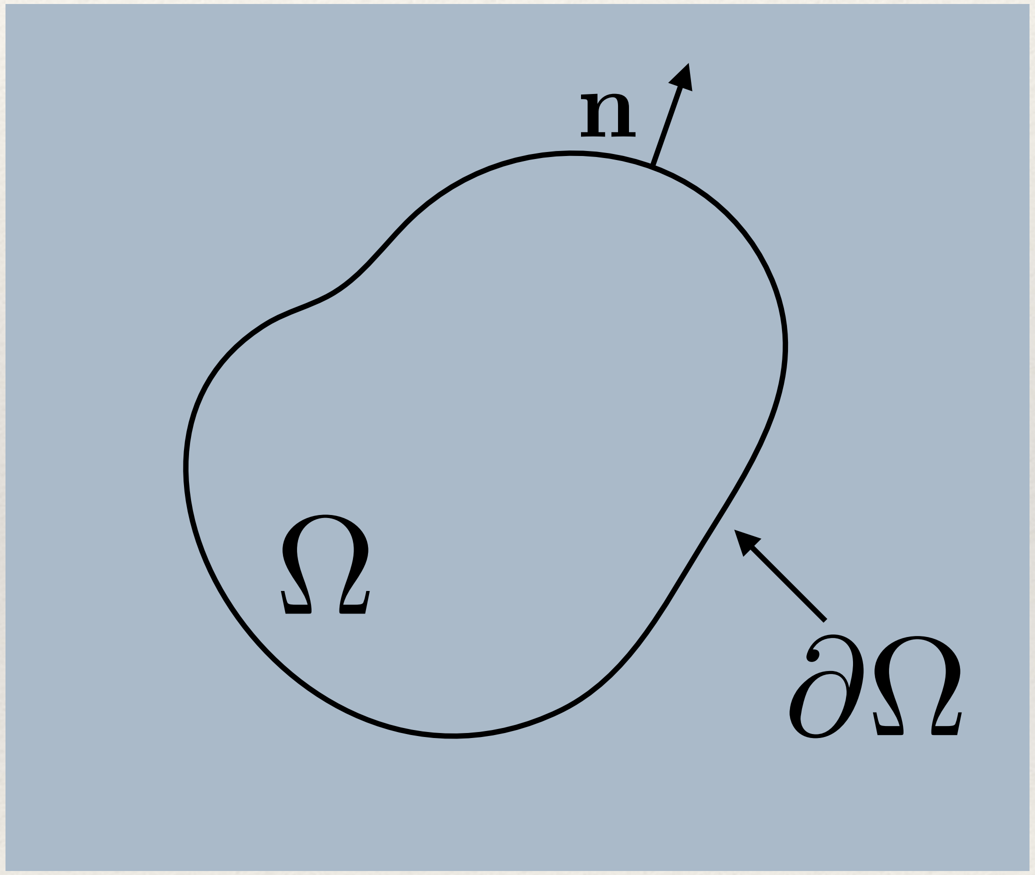
- To derive evolution equations for continua, consider arbitrary control volume Ω



Conservation Laws for Continua

- Conservation of Mass

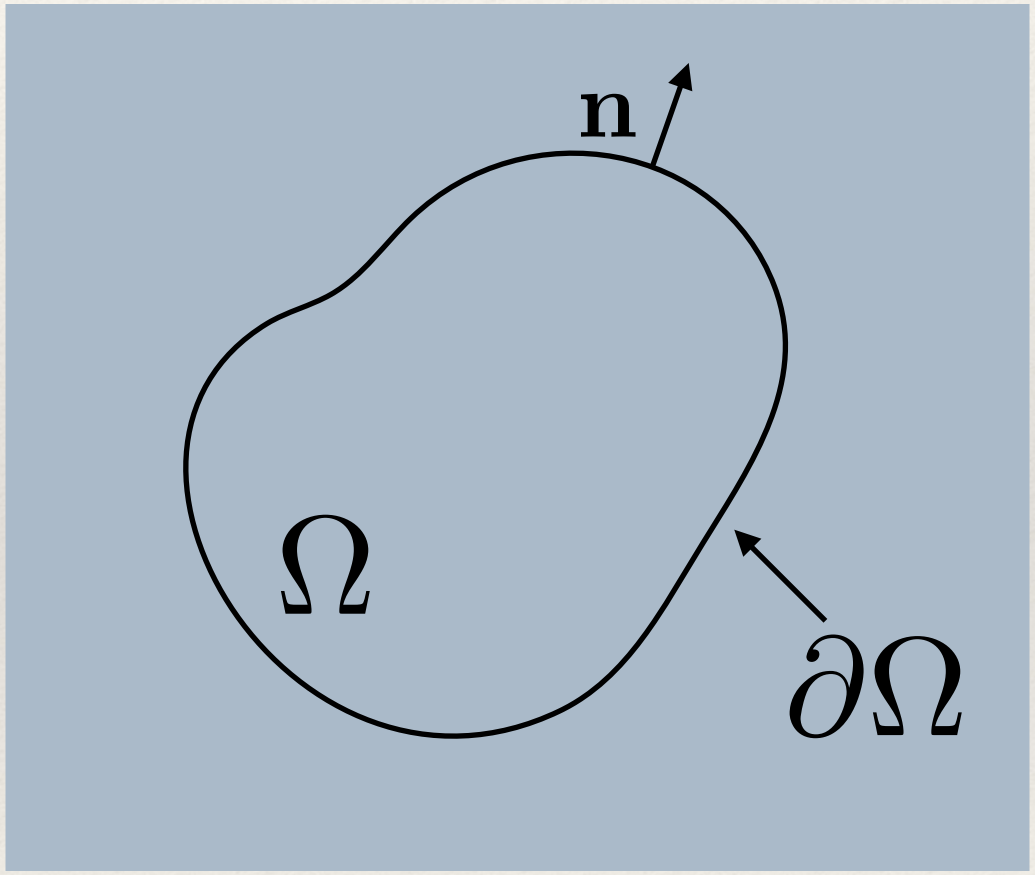
$$\int_{\Omega} \rho dV$$



Conservation Laws for Continua

- Conservation of Mass

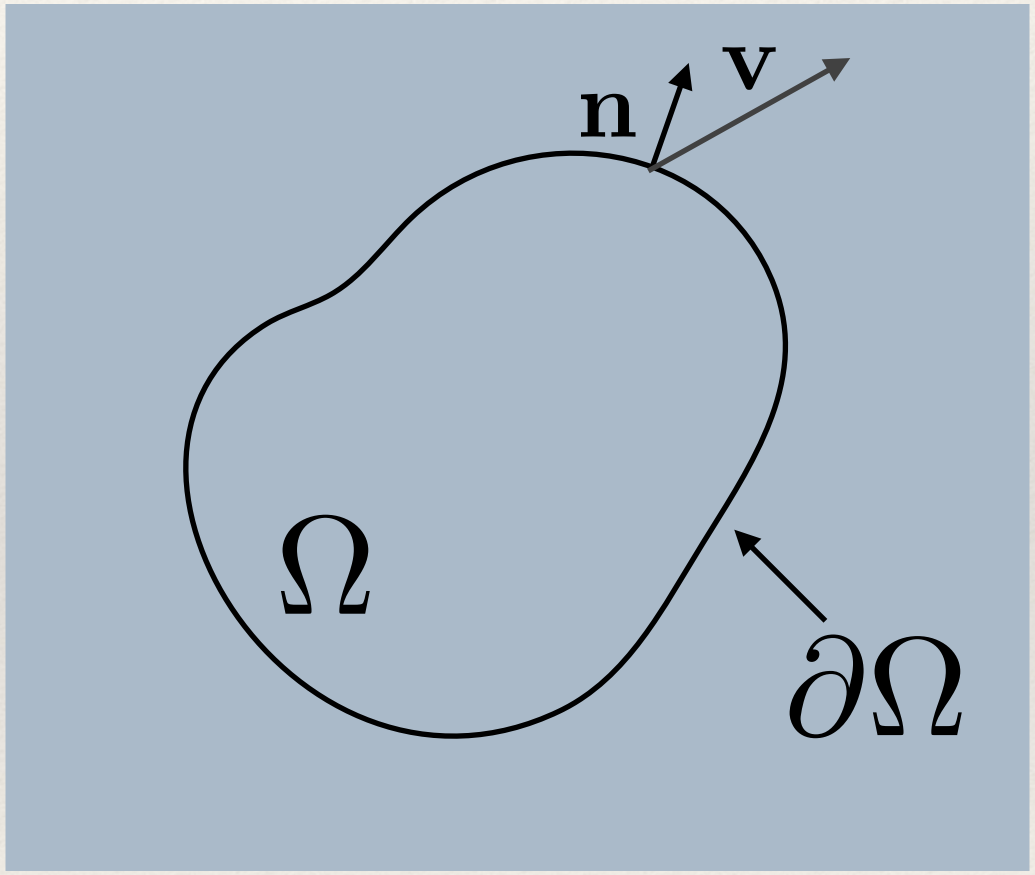
$$\frac{d}{dt}\int_{\Omega} \rho dV$$



Conservation Laws for Continua

- Conservation of Mass

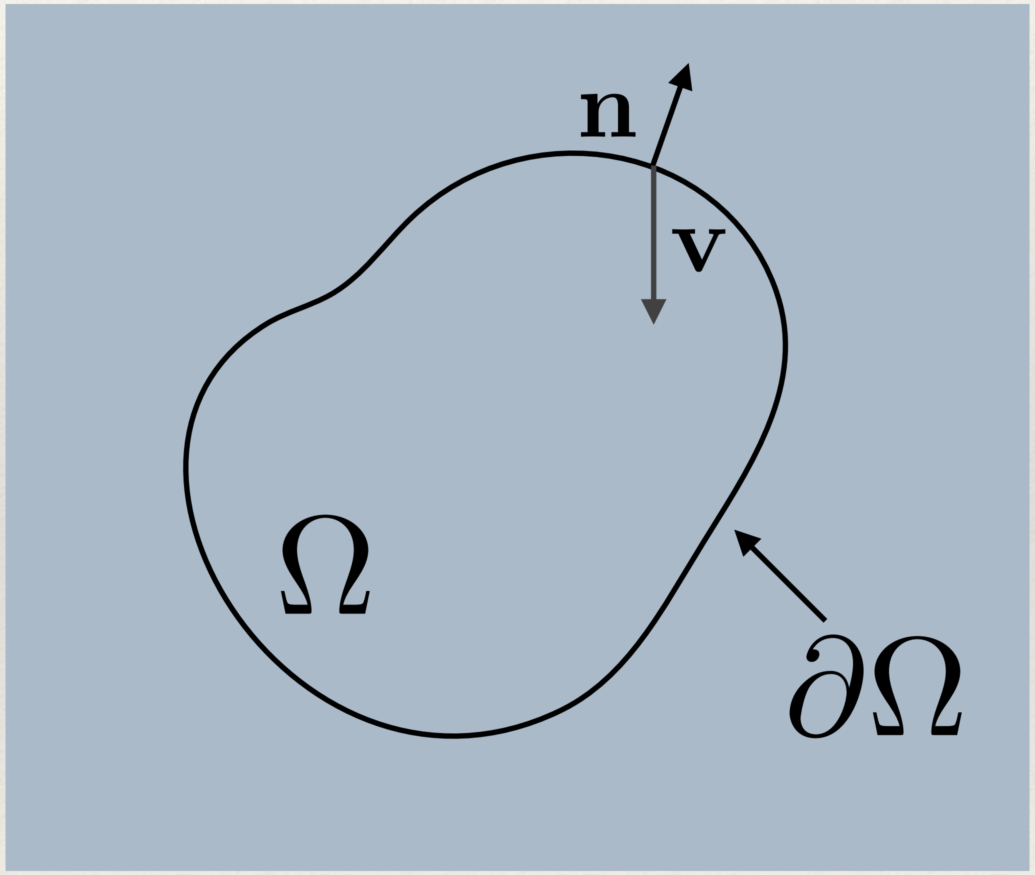
$$\frac{d}{dt}\int_{\Omega}\rho dV = -\oint_{\partial\Omega}\rho\mathbf{v}\cdot\mathbf{n}dS$$



Conservation Laws for Continua

- Conservation of Mass

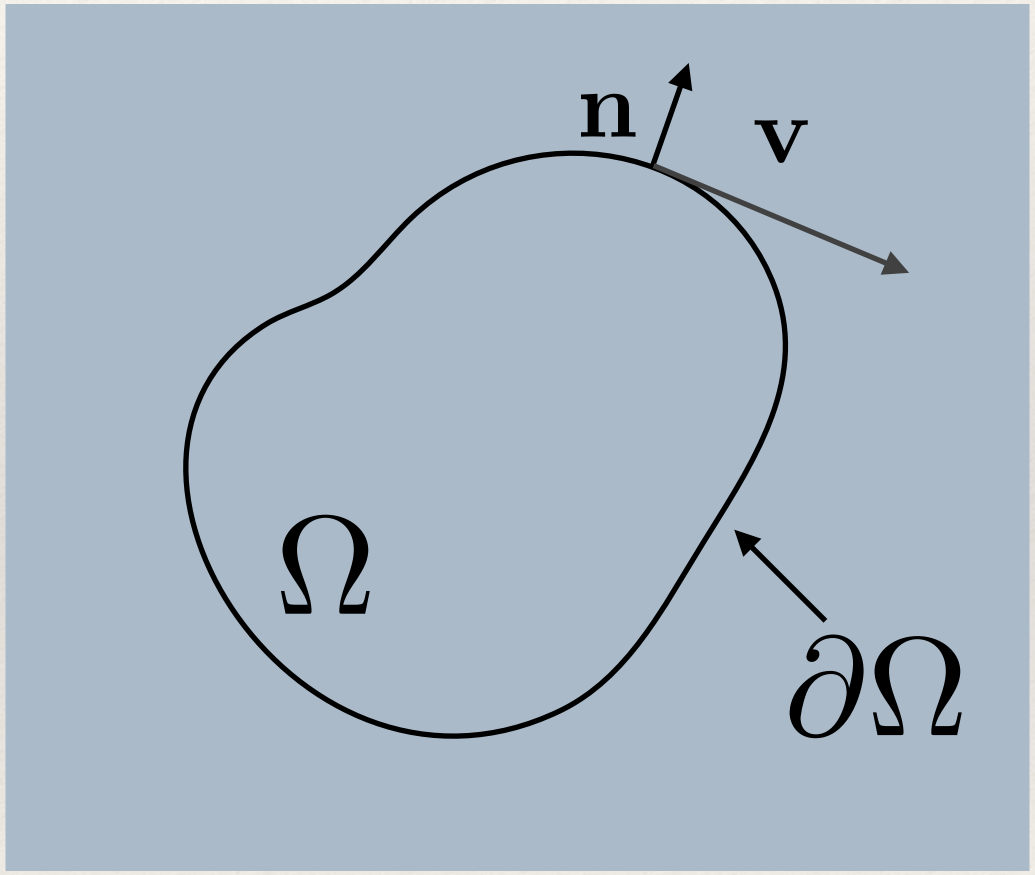
$$\frac{d}{dt}\int_{\Omega}\rho dV = -\oint_{\partial\Omega}\rho\mathbf{v}\cdot\mathbf{n}dS$$



Conservation Laws for Continua

- Conservation of Mass

$$\frac{d}{dt}\int_{\Omega} \rho dV = -\oint_{\partial\Omega} \rho \mathbf{v} \cdot \mathbf{n} dS$$



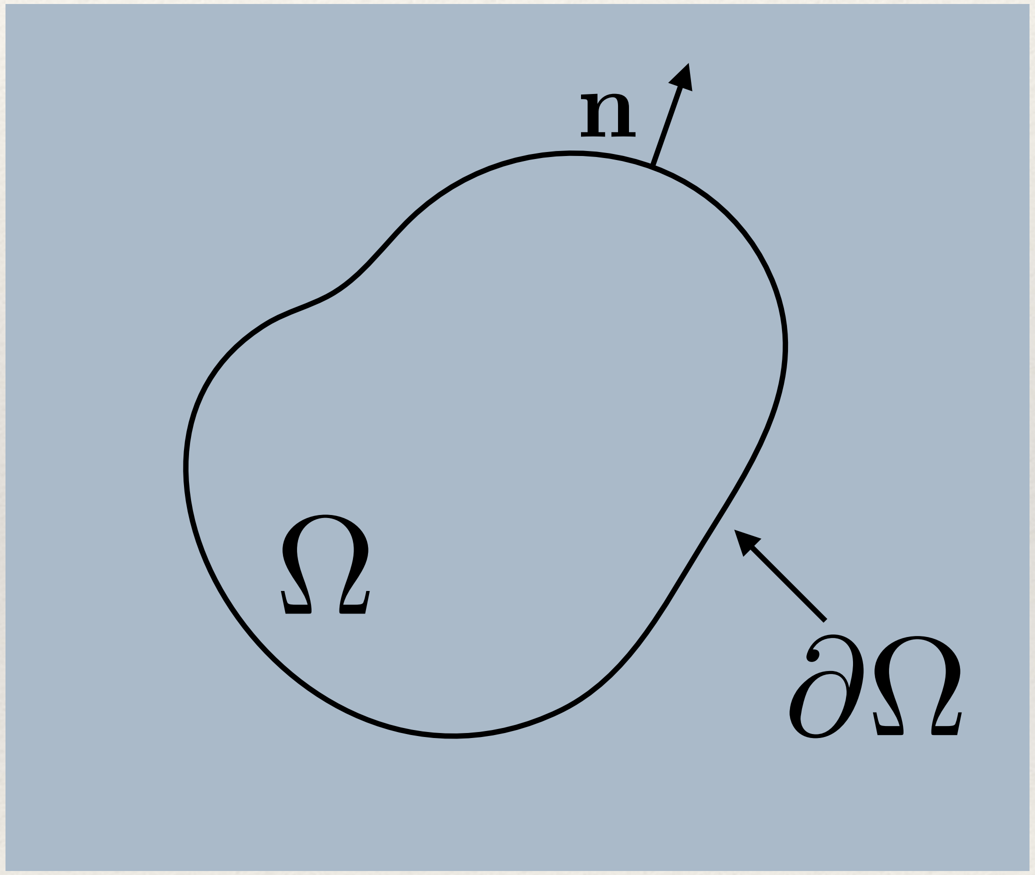
Conservation Laws for Continua

- Conservation of Mass

$$\frac{d}{dt}\int_{\Omega} \rho dV = -\oint_{\partial\Omega} \rho \mathbf{v} \cdot \mathbf{n} dS$$

- Continuity Equation

$$\Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

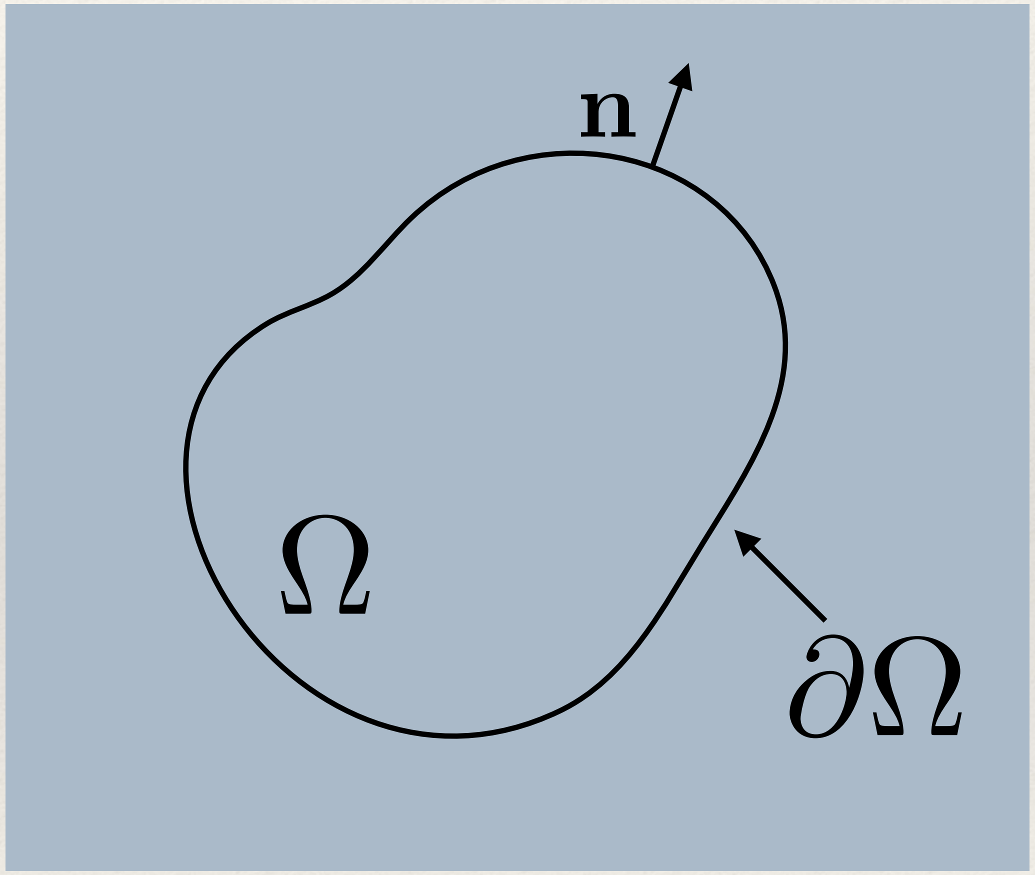


Conservation Laws for Continua

- Local conservation law

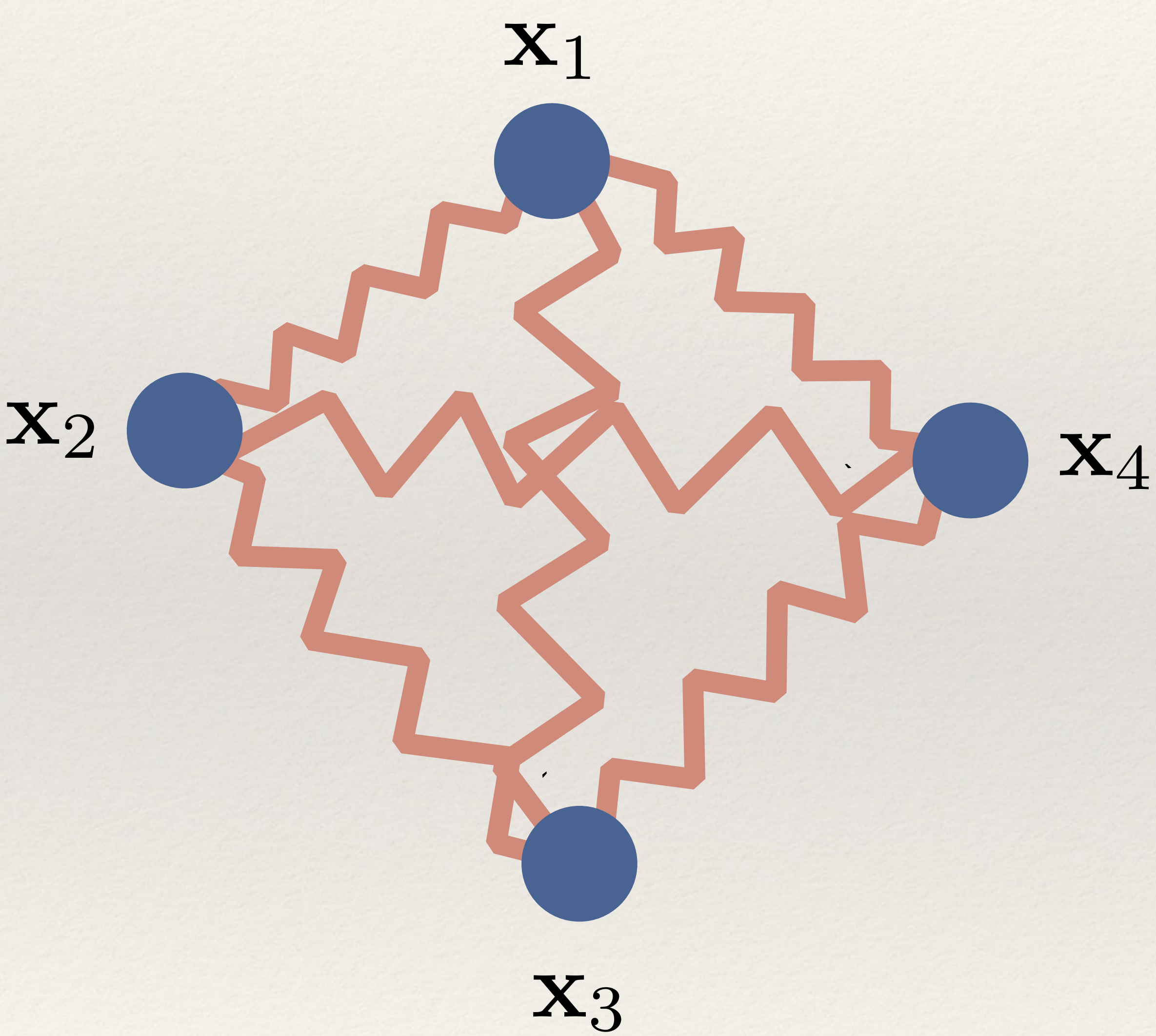
$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{f}(u) = 0$$

flux



Rigid Bodies

Rigid Body Idealization

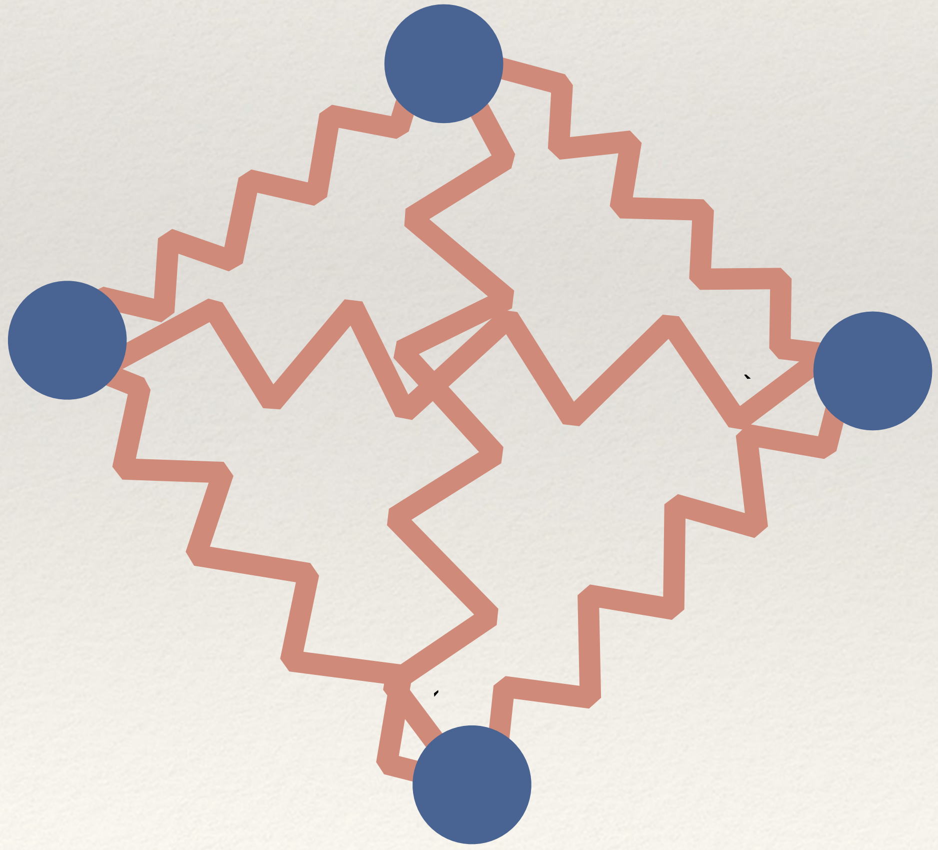


$$\mathbf{x}_i = (x_i, y_i, z_i)$$

$3 \times 4 = 12$ **degrees of freedom (DoF)**

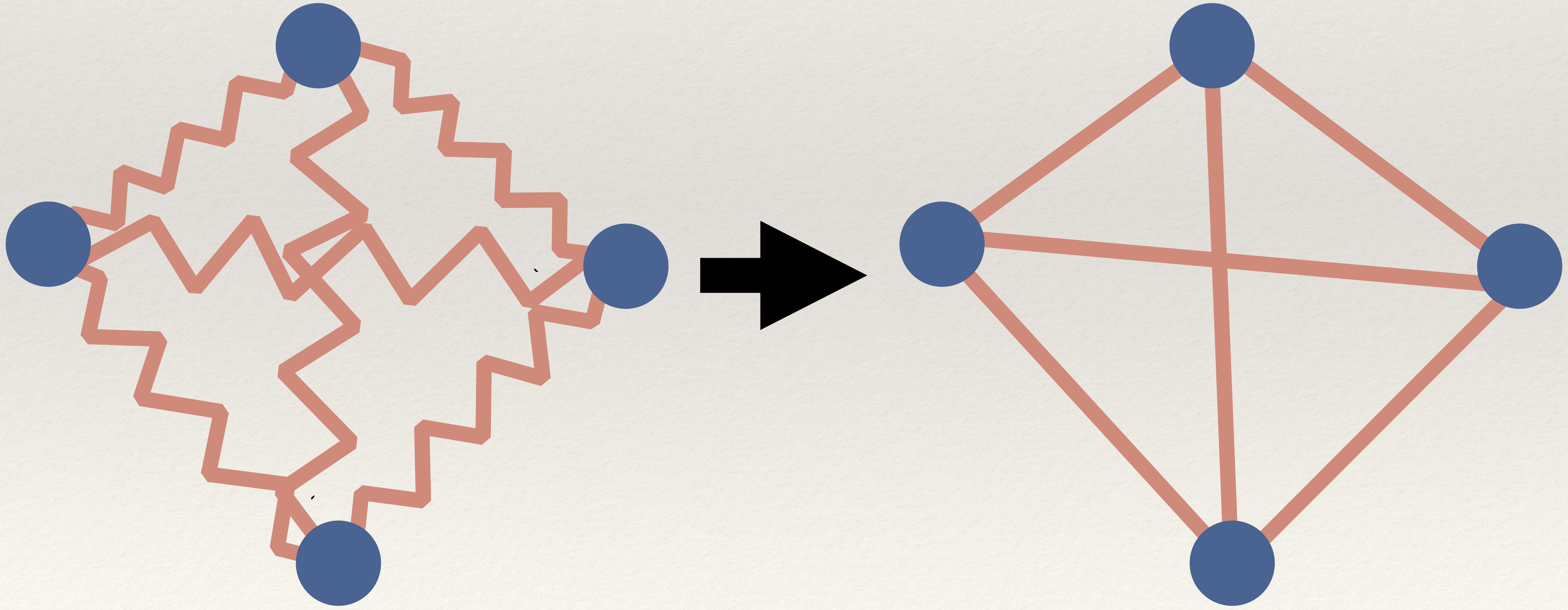
Rigid Body Idealization

- If deformation is negligible, a rigid body approximation is more efficient than a soft body model



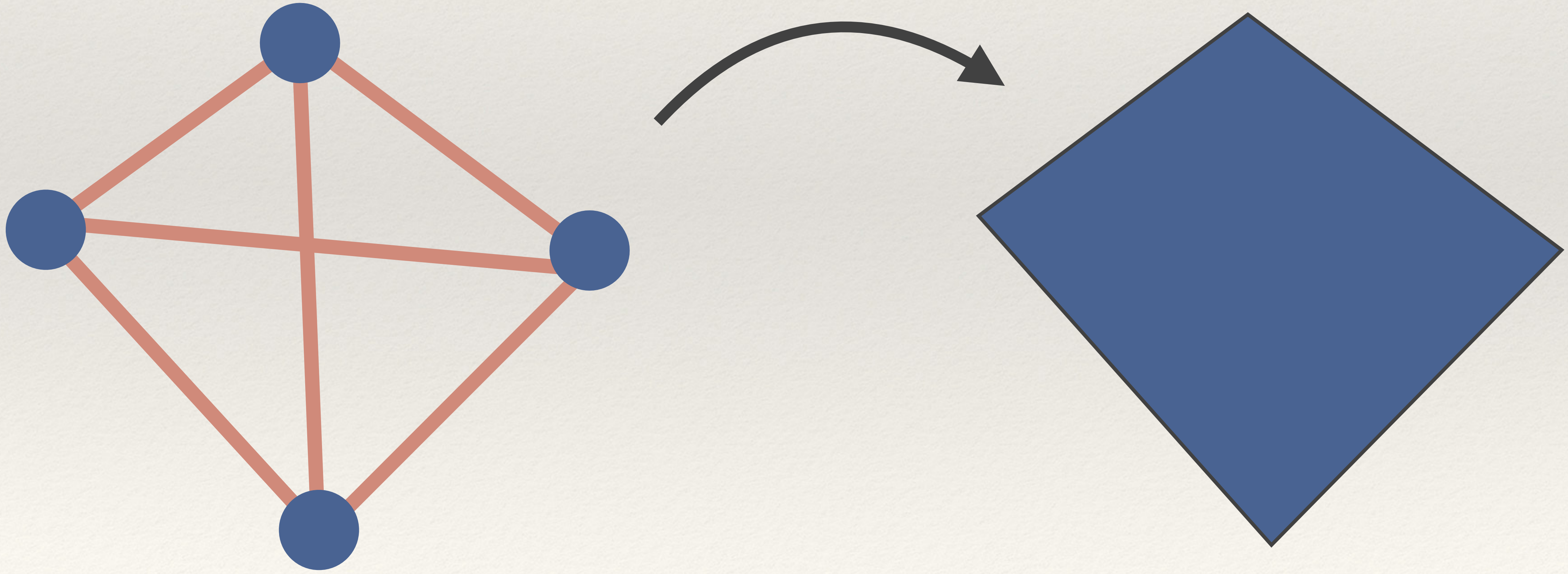
Rigid Body Idealization

- Elastic forces are replaced with constraints that particles in the body remain a fixed distance apart



Rigid Body Idealization

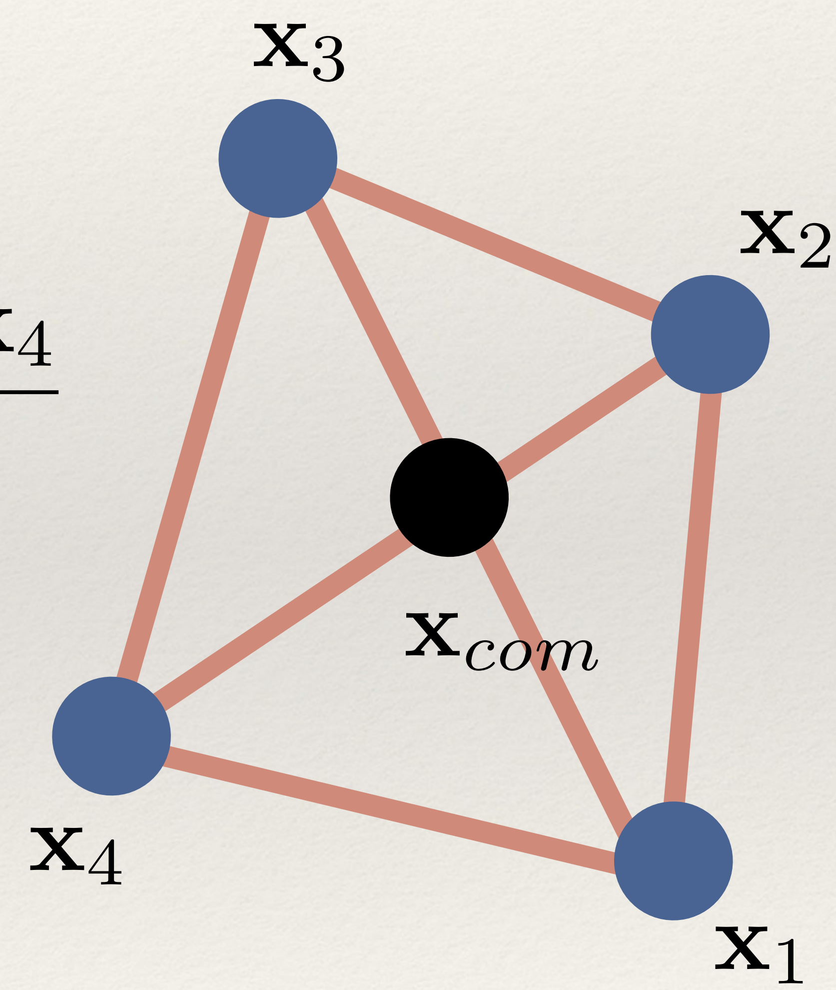
- 3n DoF are replaced with 6 DoF! position and orientation $\mathbf{x}(t), \mathbf{R}(t)$



Rigid Body Kinematics

- Center of mass

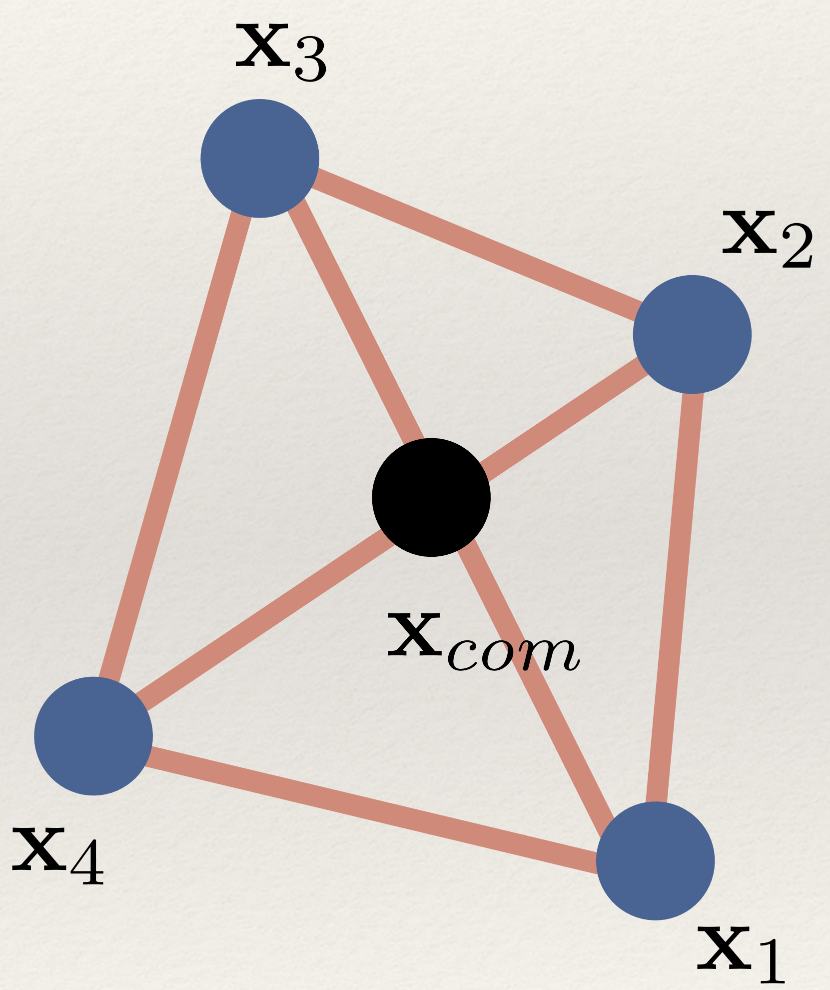
$$\mathbf{x}_{com} = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2 + m_3\mathbf{x}_3 + m_4\mathbf{x}_4}{m_1 + m_2 + m_3 + m_4}$$



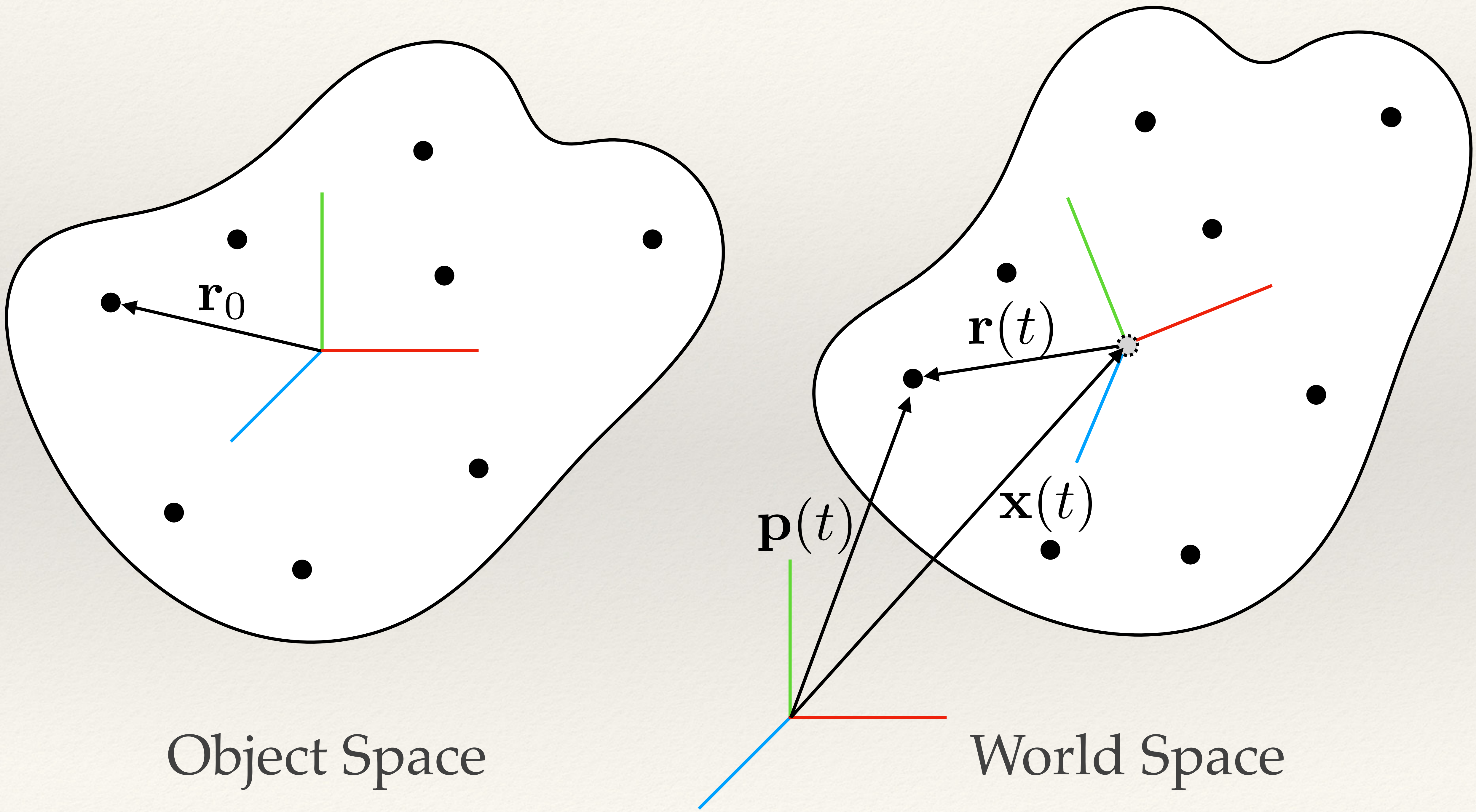
Rigid Body Kinematics

- Center of mass

$$\mathbf{x}_{com} = \frac{\sum m_i \mathbf{x}_i}{\sum m_i}$$



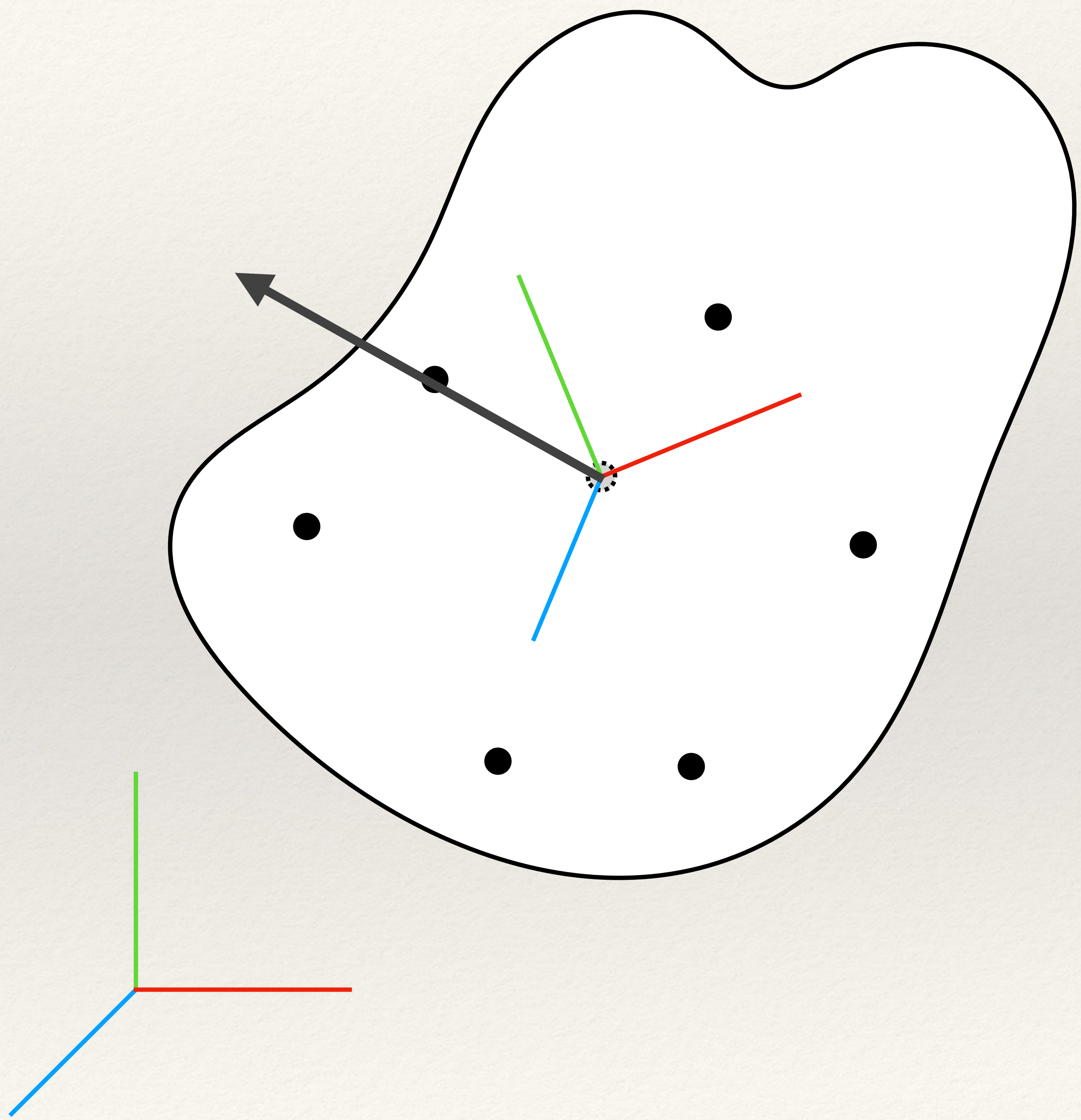
Rigid Body Coordinates



Linear and Angular Velocity

- Linear velocity

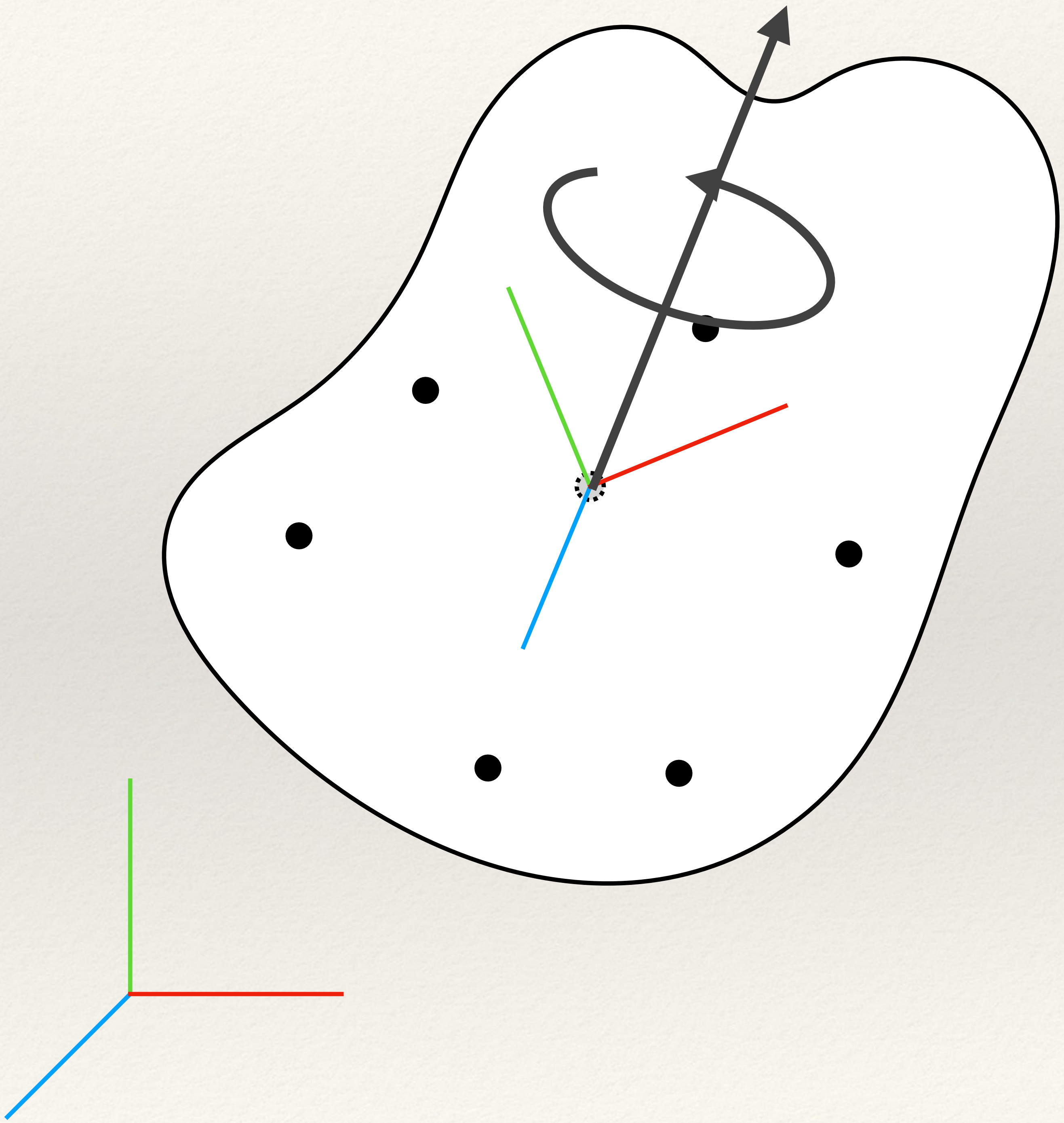
$$\mathbf{v}(t) = \dot{\mathbf{x}}(t)$$



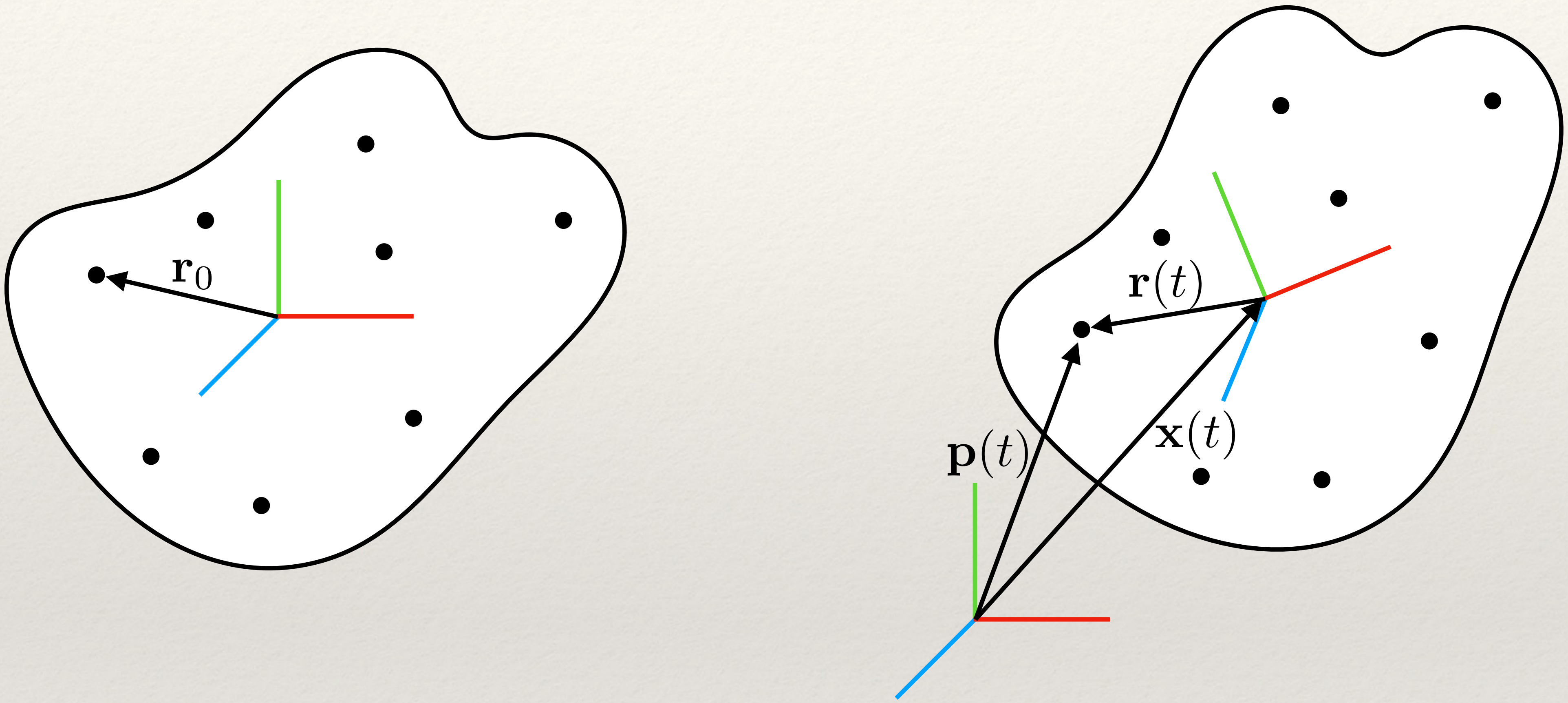
Linear and Angular Velocity

- Angular velocity

$$\omega(t)$$



Rigid Body Coordinates



particle position $$\mathbf{p}(t) = \mathbf{x}(t) + \underbrace{\mathbf{R}(t)\mathbf{r}_0}_{\mathbf{r}(t)}$$

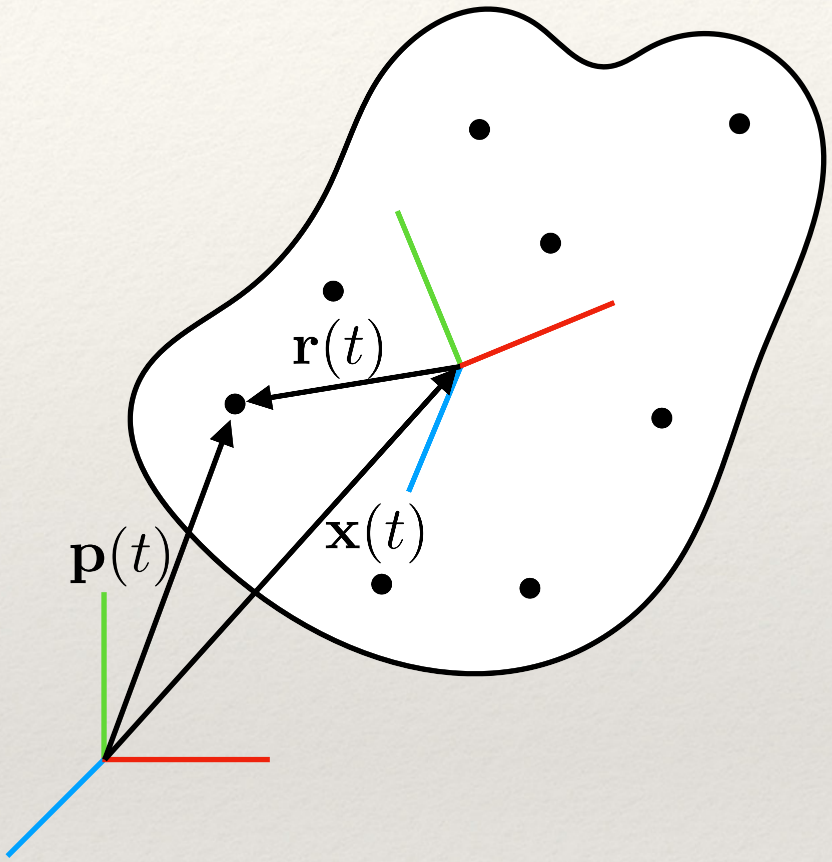
particle velocity $$\dot{\mathbf{p}}(t) = \mathbf{v}(t) + \omega(t) \times \mathbf{r}(t)$$

Linear and Angular Momentum

- Linear Momentum

$$\mathbf{P}(t) = \sum_{i=1}^{N} m_i \mathbf{v}_i(t)$$

(c.o.m. origin) $\Rightarrow \mathbf{P}(t) = m\mathbf{v}(t)$



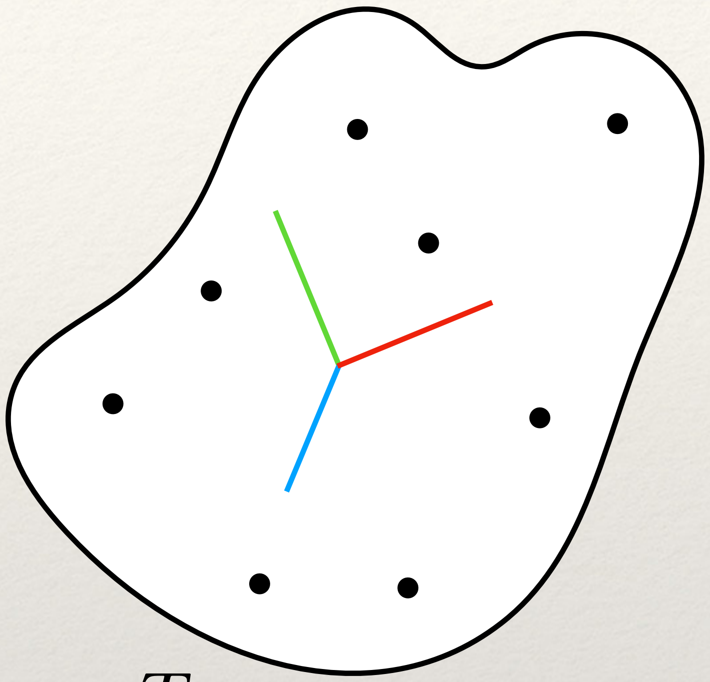
- Angular Momentum

$$\mathbf{L}(t) = \sum_{i=1}^{N} \mathbf{r}_i(t) \times m_i \mathbf{v}_i(t)$$

(c.o.m. origin) $\Rightarrow \mathbf{L}(t) = \mathbf{I}(t)\omega(t)$

$\mathbf{I}(t)$: **inertia tensor**

Rigid Body Inertia Tensor



$$\mathbf{I}(t) = \sum_{i=1}^{N} m_i \left(\mathbf{r}_i^T \mathbf{r}_i \delta - \mathbf{r}_i \mathbf{r}_i^T \right)$$

$$= \mathbf{R}(t) \underbrace{\sum_{i=1}^{N} m_i \left(\mathbf{r}_{0i}^T \mathbf{r}_{0i} \delta - \mathbf{r}_{0i} \mathbf{r}_{0i}^T \right)}_{\mathbf{I}_0} \mathbf{R}(t)^T$$

$$= \mathbf{R}(t) \mathbf{I}_0 \mathbf{R}(t)^T$$

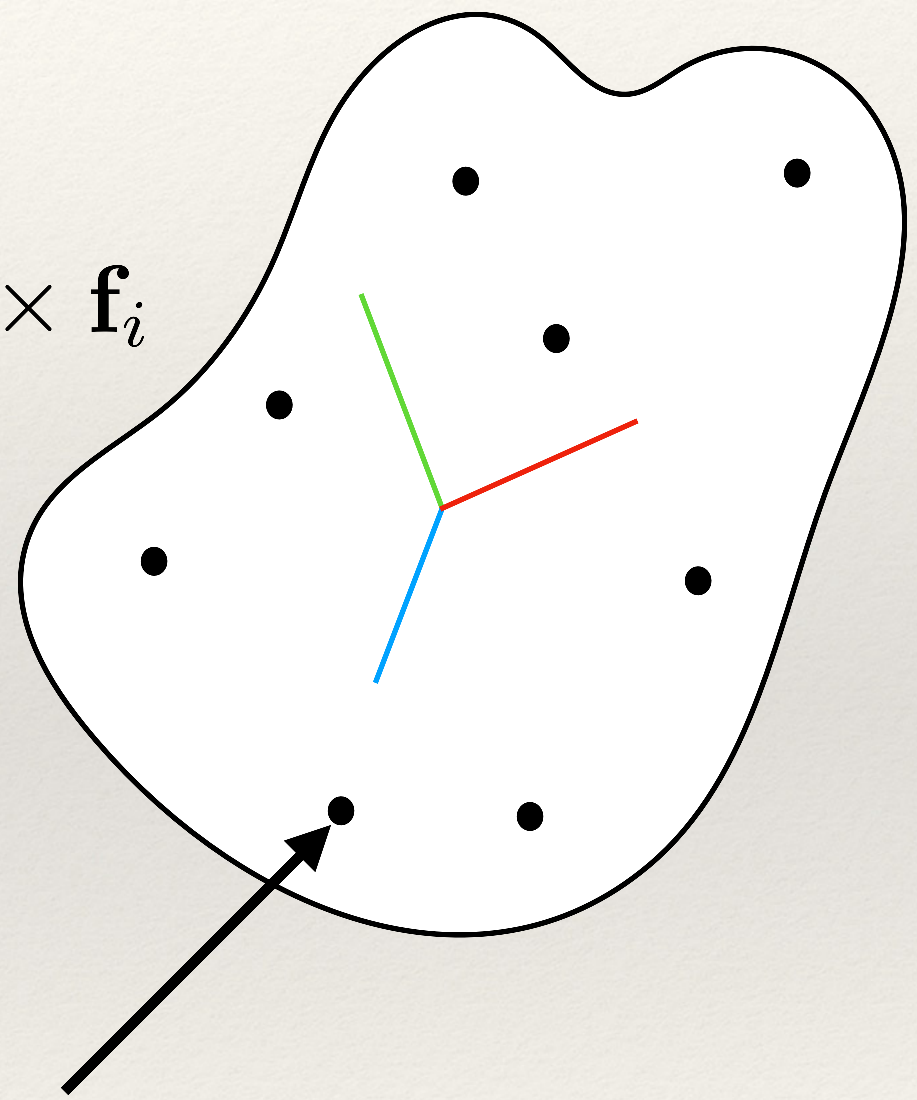
Linear and Angular Momentum

- No net force =>
  - linear momentum and velocity constant
- No net torque =>
  - angular momentum constant
  - angular velocity not necessarily constant

$$\mathbf{P}(t) = m\mathbf{v}(t)$$
$$\mathbf{L}(t) = \mathbf{I}(t)\omega(t)$$

Newton-Euler Equations for Rigid Bodies

$$\frac{d}{dt}\begin{pmatrix}\mathbf{P}(t)\\ \mathbf{L}(t)\end{pmatrix} = \begin{pmatrix}\mathbf{f}(t)\\ \tau(t)\end{pmatrix}, \qquad \begin{aligned}\mathbf{f}(t) &= \sum \mathbf{f}_i\\ \tau(t) &= \sum \mathbf{r}_i \times \mathbf{f}_i\end{aligned}$$



- Summary

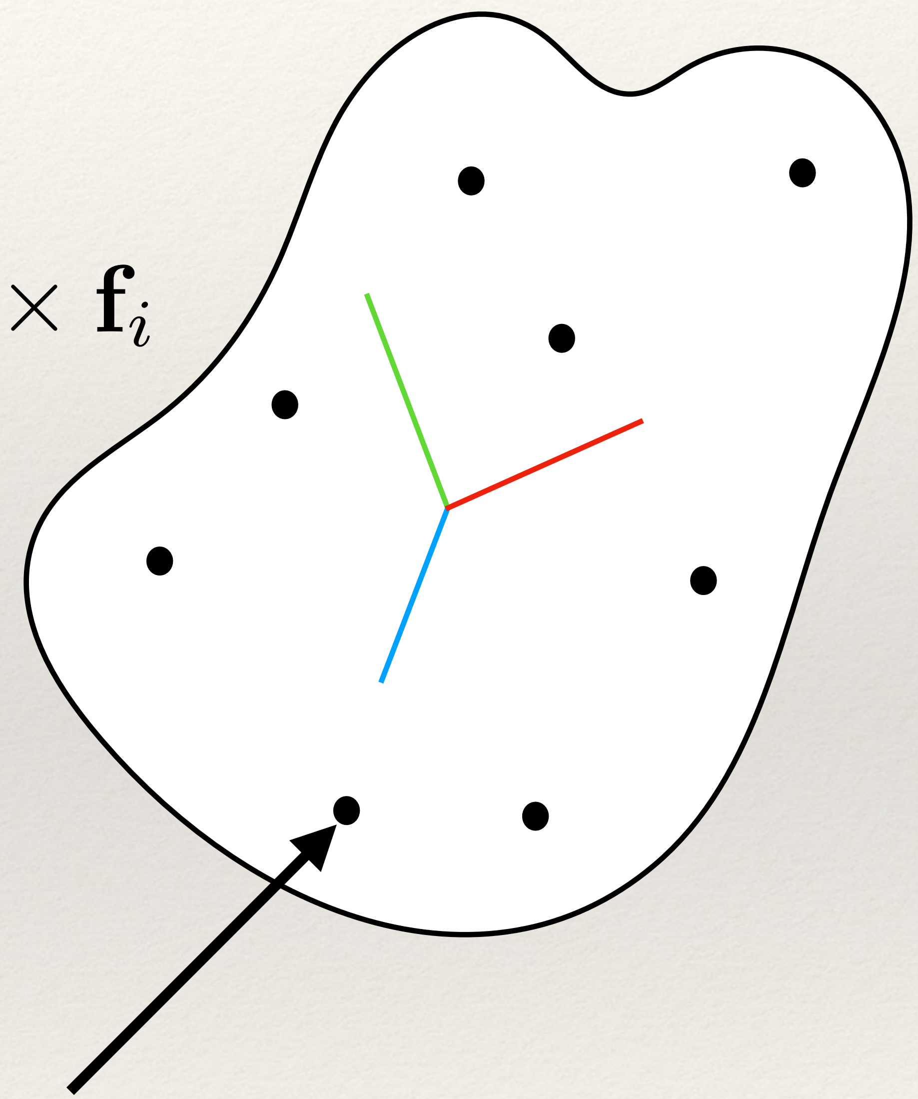
$$\frac{d}{dt}\begin{pmatrix}\mathbf{x}(t)\\ R(t)\\ \mathbf{P}(t)\\ \mathbf{L}(t)\end{pmatrix} = \begin{pmatrix}\mathbf{v}(t)\\ \omega^\star(t)R(t)\\ \mathbf{f}(t)\\ \tau(t)\end{pmatrix}$$

Newton-Euler Equations for Rigid Bodies

```
struct RigidBody {
  double mass;
  Eigen::Matrix3d I0;
  Eigen::Vector3d pos, P, frc;
  Eigen::Matrix3d R;
  Eigen::Vector3d L, trq;
};
```

$$= \sum \mathbf{f}_i$$

$$= \sum \mathbf{r}_i \times \mathbf{f}_i$$



$$\frac{d}{dt}\begin{pmatrix} \mathbf{x}(t) \\ R(t) \\ \mathbf{P}(t) \\ \mathbf{L}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{v}(t) \\ \omega^{\star}(t)R(t) \\ \mathbf{f}(t) \\ \tau(t) \end{pmatrix}$$

Soft Bodies

Adding Elasticity and Damping

$$\mathbf{f} = m\mathbf{a}$$

$$m\mathbf{a} = \mathbf{f}$$

$$\mathbf{K}(\mathbf{x} - \mathbf{u}) + \mathbf{D}(\mathbf{v}) + \mathbf{M}\mathbf{a} = \mathbf{f}_{ext}$$

in terms of displacements $\mathbf{d}$

$$\mathbf{K}(\mathbf{d}) + \mathbf{D}(\dot{\mathbf{d}}) + \mathbf{M}\ddot{\mathbf{d}} = \mathbf{f}_{ext}$$

K

- Generalization of spring stiffness, called *stiffness* matrix
- Sparse, Symmetric, Diagonally Dominant, Row/Col sums 0
- Positive semi-definite
- Negative Jacobian of forces
- Negative Hessian of energy

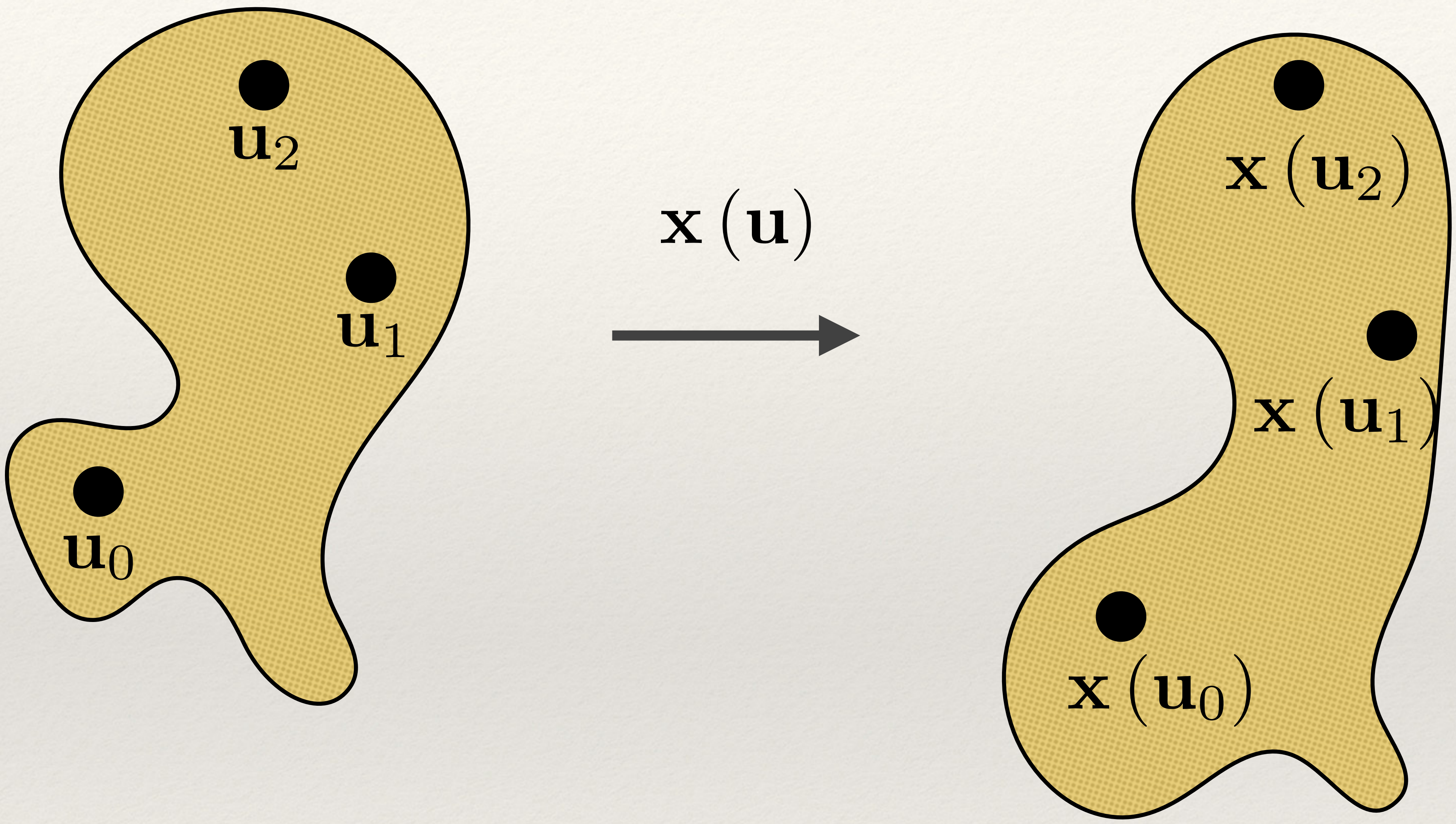
$$\mathbf{K} = \frac{-\partial \mathbf{f}_i}{\partial \mathbf{x}_j} = \frac{-\partial \eta}{\partial \mathbf{x}_i \partial \mathbf{x}_j}$$

So how do we calculate elastic forces?

Elastic Forces Cookbook

- Deformation Function (world pos, rest position)
- Deformation Gradient (deformation function)
- Strain (deformation gradient)
- Stress (strain)
- Energy (stress, strain)
- Forces (energy)

Deformation and Its Gradient



$$\mathbf{x}(\mathbf{u}) = \mathbf{x}(\mathbf{u}_0) + \mathbf{F}\left(\mathbf{u} - \mathbf{u}_0\right)$$

$$\frac{\partial \mathbf{x}}{\partial \mathbf{u}} = \mathbf{F}$$

Strain is

- a measure of deformation
- dimensionless (i.e. has no units)
- a function of the deformation gradient

Strain Metrics

- Green's finite $\epsilon = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right)$

- Cauchy's infinitesimal $\epsilon = \frac{1}{2}\left(\mathbf{F}^T + \mathbf{F}\right) - \mathbf{I}$

- Co-rotated $\epsilon = \frac{1}{2}\left(\tilde{\mathbf{F}}^T + \tilde{\mathbf{F}}\right) - \mathbf{I}$

  where $\mathbf{F} = \mathbf{Q}\tilde{\mathbf{F}}$

Stress

- stress is a function of strain
- stress has units of (Newton / meter^2)
- models of stress-strain relationships can be highly complex, especially with organic materials

Linear Stress-strain Relationships

General, linear

$$\sigma = \mathbf{C}\epsilon$$

isotropic material

$$\sigma = \lambda \mathrm{Tr}\left(\epsilon\right)\mathbf{I} + 2\mu\epsilon$$

Elastic Potential, Traction, Force

$$\eta = \frac{1}{2}\sigma : \epsilon = \frac{1}{2}\sum_{i,j}\sigma_{ij}\epsilon_{ij}$$

$$\tau = \sigma\mathbf{n}$$

$$\mathbf{f}_i = -\frac{\partial\eta}{\partial\mathbf{x}_i}$$

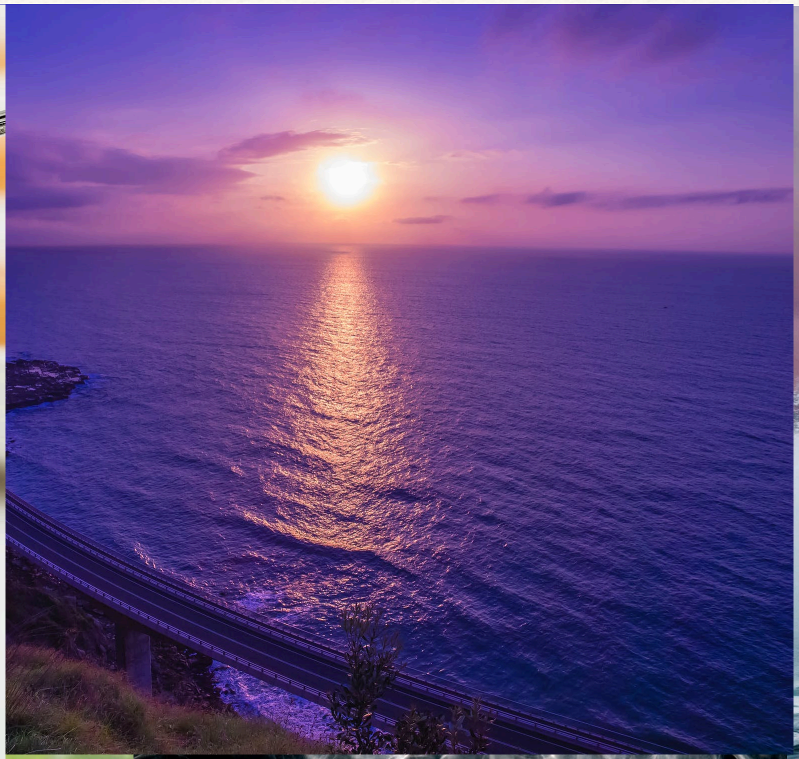
$$\mathbf{f} = \oint_{\partial R} \sigma\mathbf{n}\, dS$$

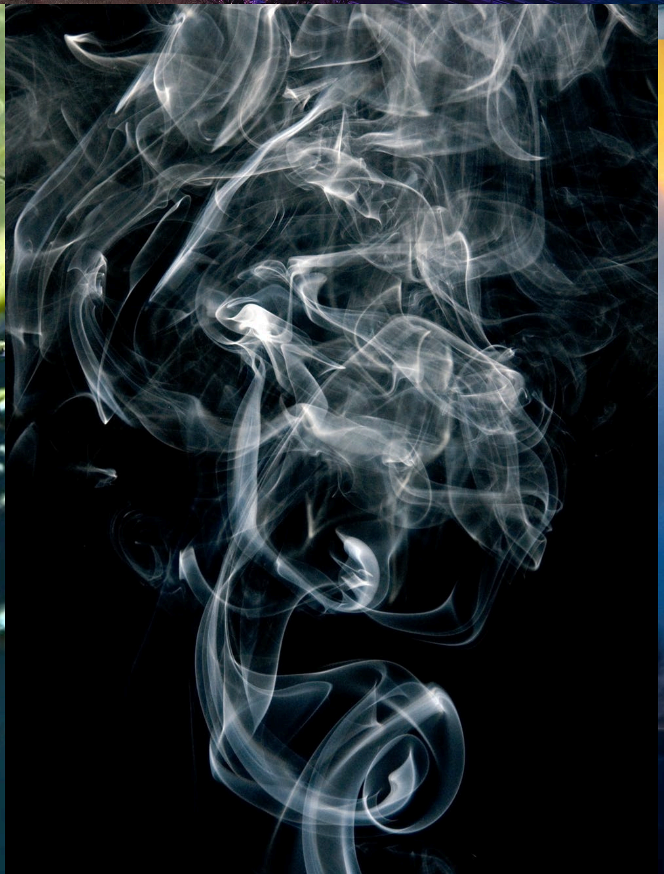
Plasticity

$$\mathbf{F} = \mathbf{F}_e \mathbf{F}_p$$

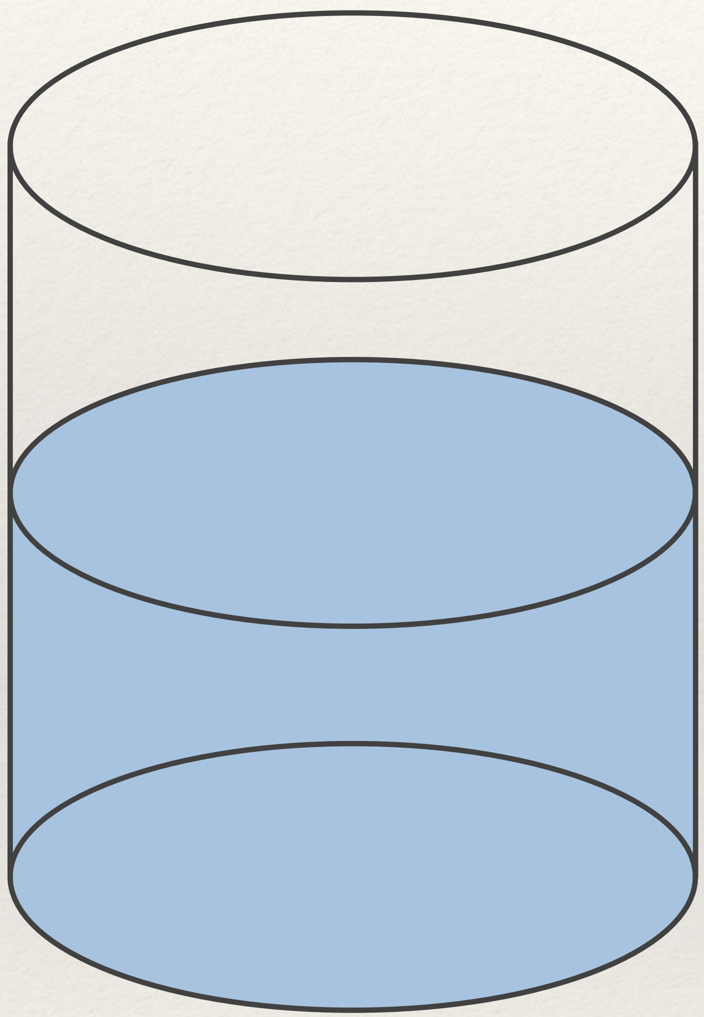
Fluids

Fluids

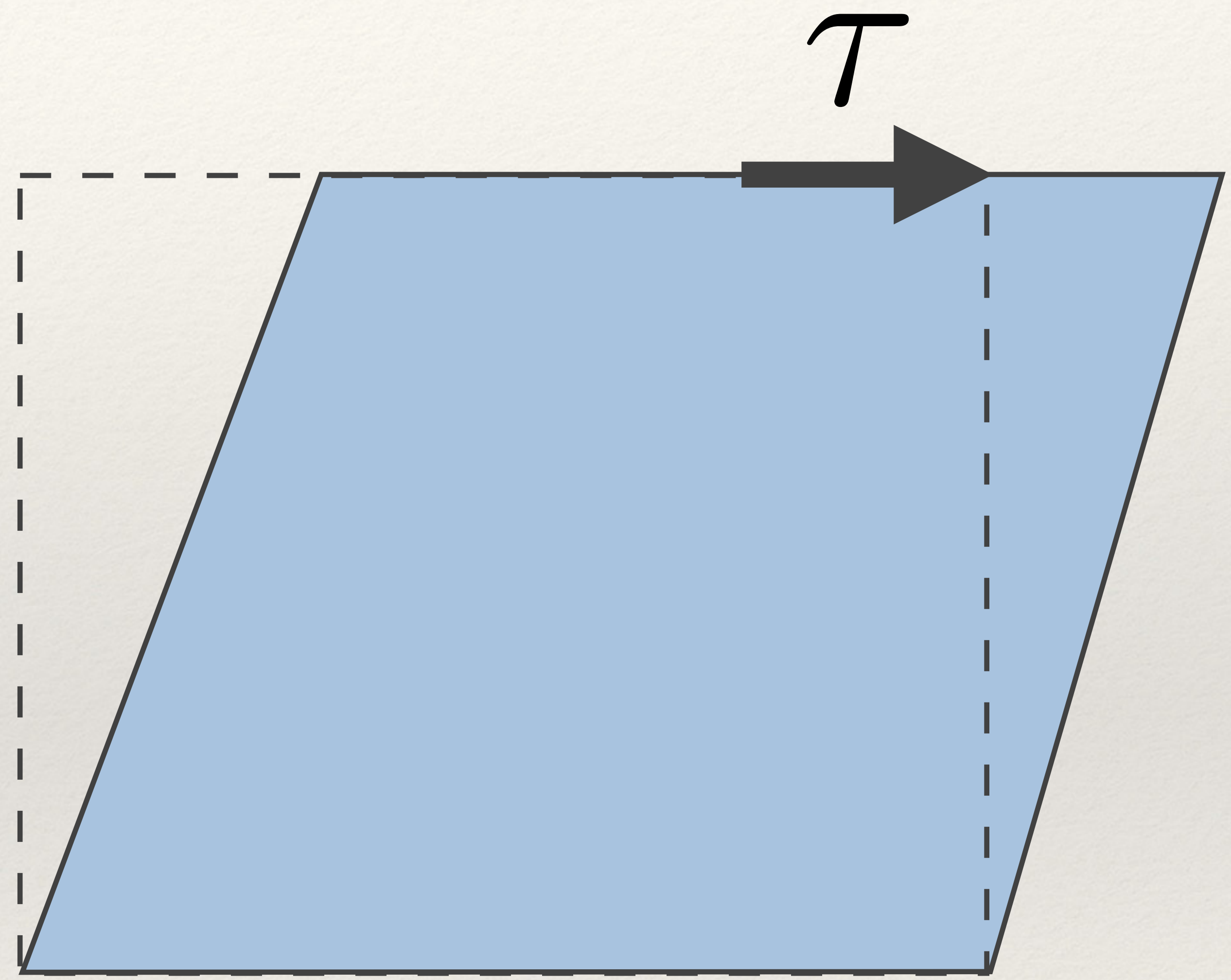




Fluids

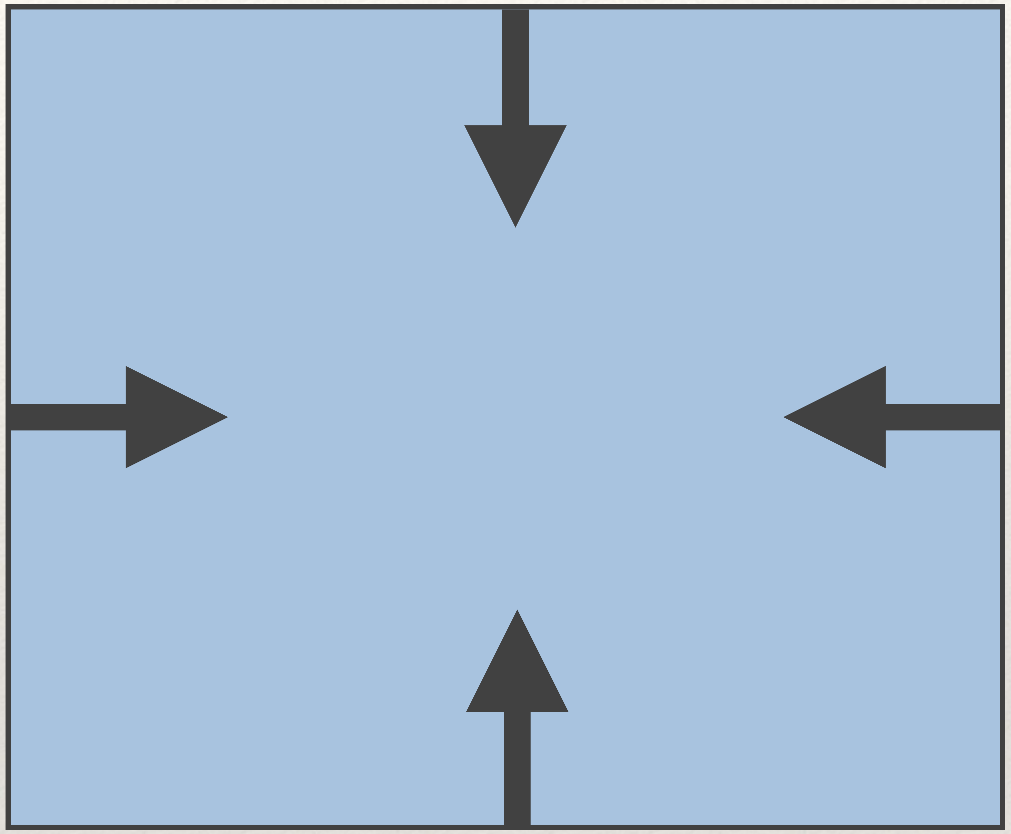


take shape
of container



can’t support
shear stress

Fluids



can support
normal stress

Navier-Stokes Equations

$$\rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \mu \Delta \mathbf{u} + \mathbf{f}$$

$$\nabla \cdot \mathbf{u} = 0$$

ρ : density p : pressure $\mathbf{f}$: forces

$\mathbf{u}$: velocity μ : viscosity

Navier-Stokes Equations

$$\overset{\text{m}}{\rho}\,\overbrace{(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u})}^{\mathbf{a}} = -\overbrace{\nabla p + \mu\Delta\mathbf{u} + \mathbf{f}}^{\mathbf{f}}$$

$$\nabla\cdot\mathbf{u} = 0$$

ρ : density $\quad$ p : pressure $\quad$ $\mathbf{f}$: forces

$\mathbf{u}$: velocity $\quad$ μ : viscosity

Navier-Stokes Equations

$$\overset{\mathrm{m}}{\rho}\,\overbrace{(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u})}^{\mathbf{a}} = \overbrace{-\nabla p + \mu\Delta\mathbf{u} + \mathbf{f}}^{\mathbf{f}}$$

$$\nabla\cdot\mathbf{u} = 0$$

ρ : density

p : pressure

$\mathbf{f}$: forces

$\mathbf{u}$: velocity

μ : viscosity

Navier-Stokes Equations

$$\overset{\text{m}}{\rho}\,\overbrace{(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u})}^{\mathbf{a}} = \overbrace{-\nabla p + \mu\Delta\mathbf{u} + \mathbf{f}}^{\mathbf{f}}$$

$$\nabla\cdot\mathbf{u} = 0$$

ρ : density p : pressure $\mathbf{f}$: forces

$\mathbf{u}$: velocity μ : viscosity

Navier-Stokes Equations

$$\overset{\text{m}}{\rho}\,\overbrace{\left(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}\right)}^{\mathbf{a}} = -\overbrace{\nabla p + \mu\Delta\mathbf{u} + \mathbf{f}}^{\mathbf{f}}$$

$$\nabla\cdot\mathbf{u} = 0$$

ρ : density

p : pressure

$\mathbf{f}$: forces

$\mathbf{u}$: velocity

μ : viscosity

Navier-Stokes Equations

$$\overset{\text{m}}{\rho}\,\overbrace{\left(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}\right)}^{\mathbf{a}} = -\overbrace{\nabla p + \mu\Delta\mathbf{u} + \mathbf{f}}^{\mathbf{f}}$$

$$\nabla\cdot\mathbf{u} = 0$$

ρ : density $\quad$ p : pressure $\quad$ $\mathbf{f}$: forces

$\mathbf{u}$: velocity $\quad$ μ : viscosity

Navier-Stokes Equations

$$\overset{\text{m}}{\rho}\,\overbrace{\left(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}\right)}^{\mathbf{a}} = -\overbrace{\nabla p + \mu\Delta\mathbf{u} + \mathbf{f}}^{\mathbf{f}}$$

$$\nabla\cdot\mathbf{u} = 0$$

ρ : density

p : pressure

$\mathbf{f}$: forces

$\mathbf{u}$: velocity

μ : viscosity

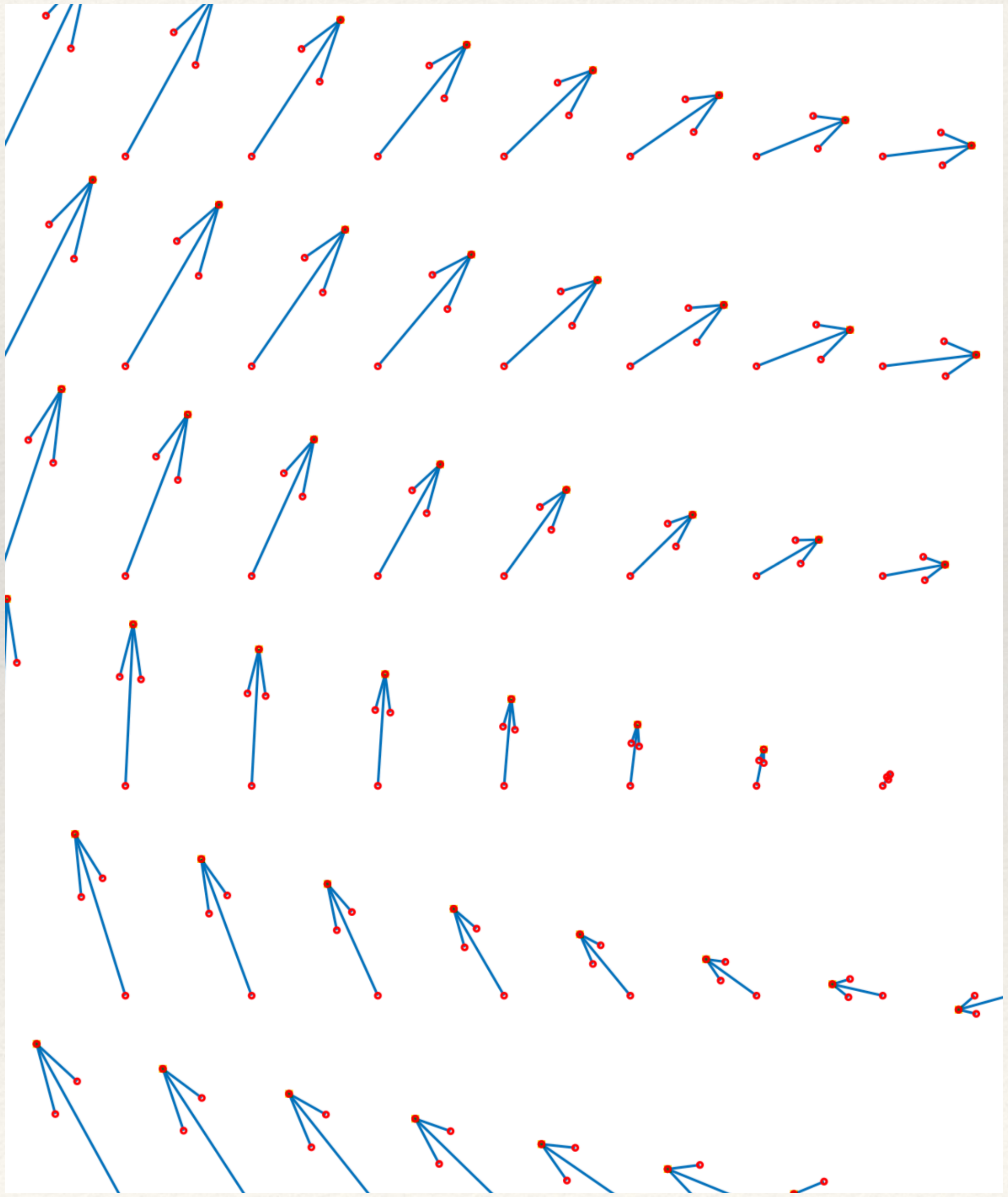
Material Derivative

$$\overbrace{\rho}^{\mathrm{m}} \overbrace{\left(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}\right)}^{\mathbf{a}} = -\overbrace{\nabla p + \mu \Delta \mathbf{u} + \mathbf{f}}^{\mathbf{f}}$$

$$\nabla \cdot \mathbf{u} = 0$$

Material Derivative

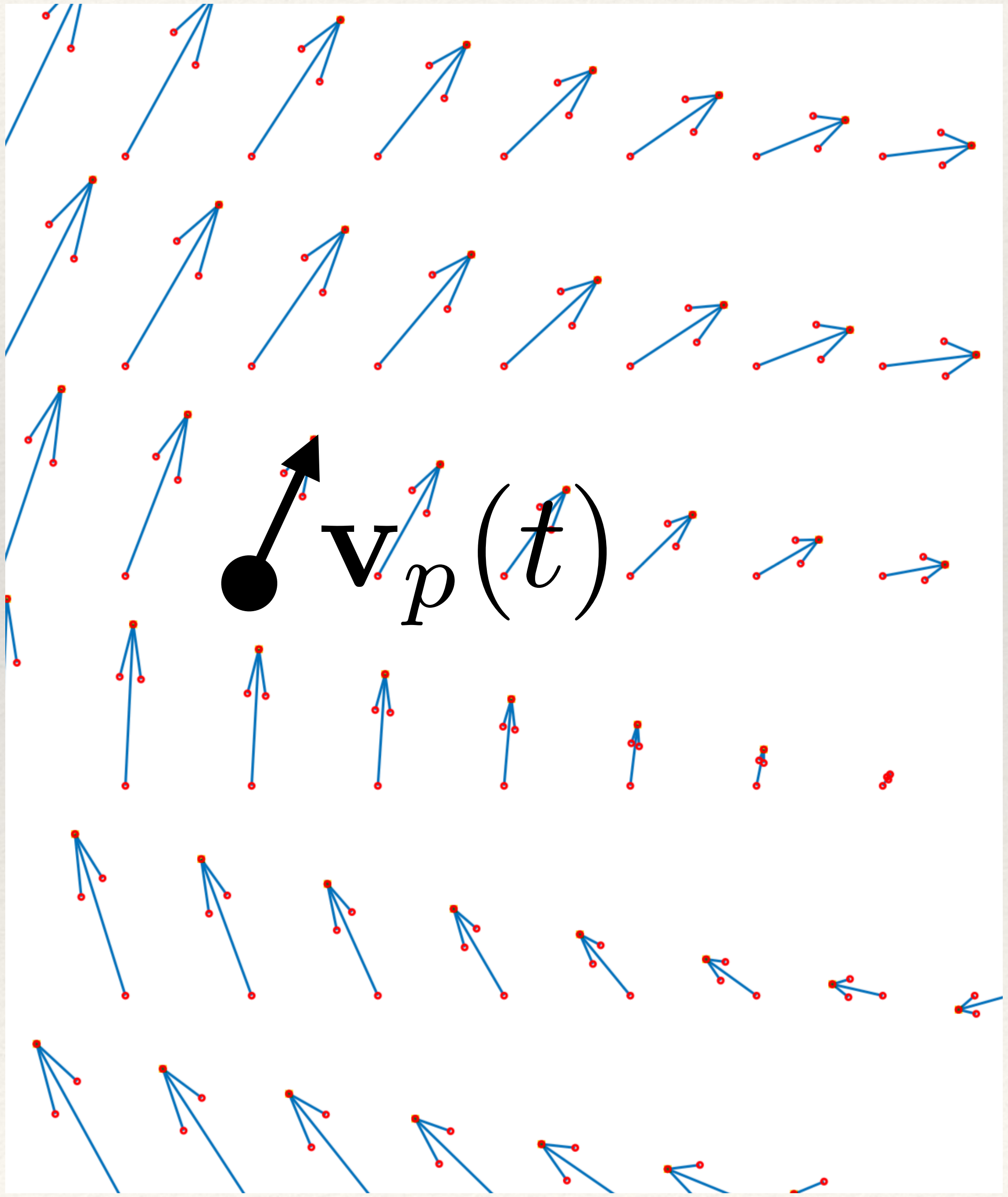
$$\mathbf{u}(\mathbf{x}, t)$$



Material Derivative

$$\mathbf{a}_p(t) = \frac{d}{dt}\mathbf{v}_p(t)$$

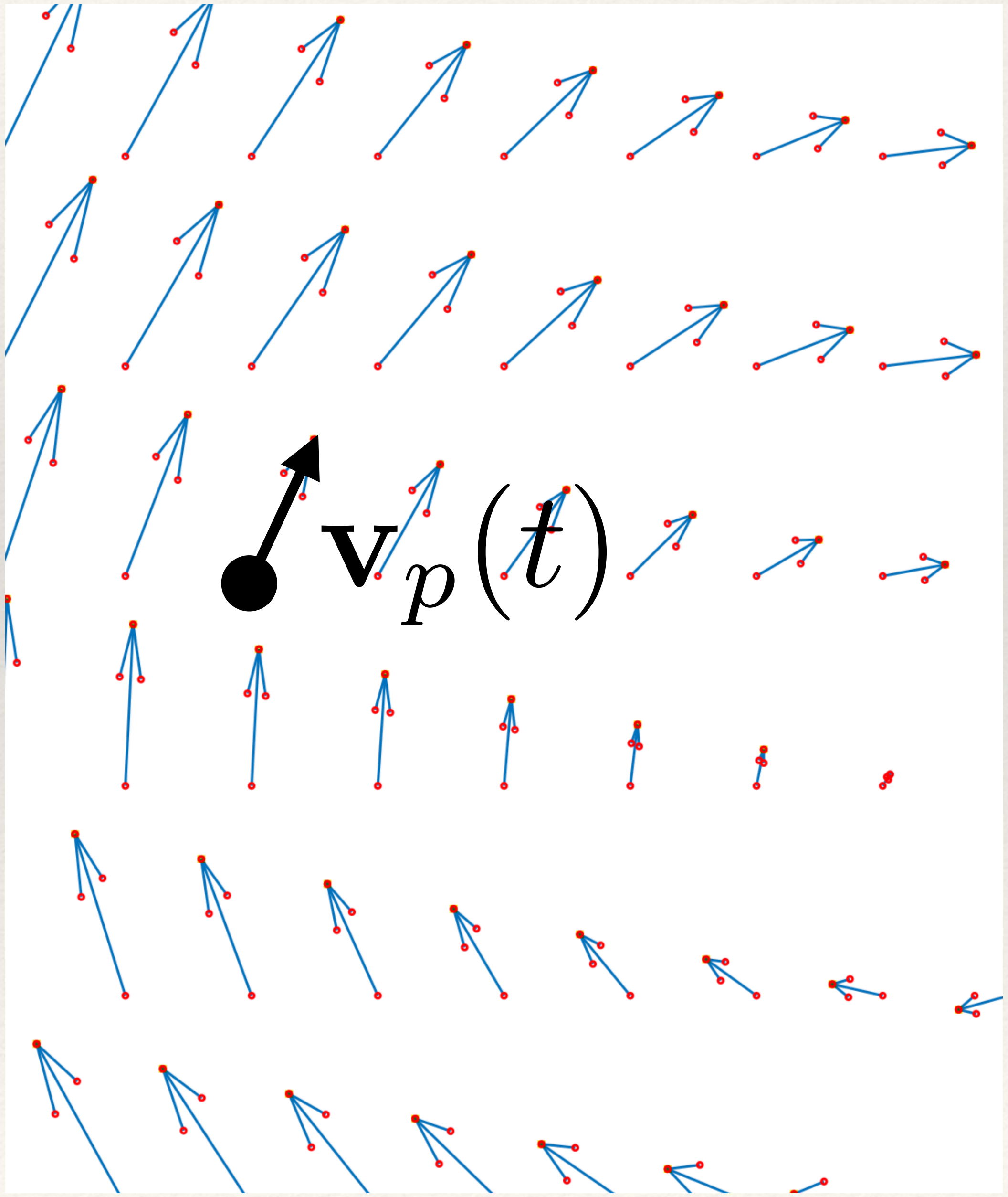
$\mathbf{u}(\mathbf{x}, t)$



Material Derivative

$$\mathbf{u}(\mathbf{x}, t)$$

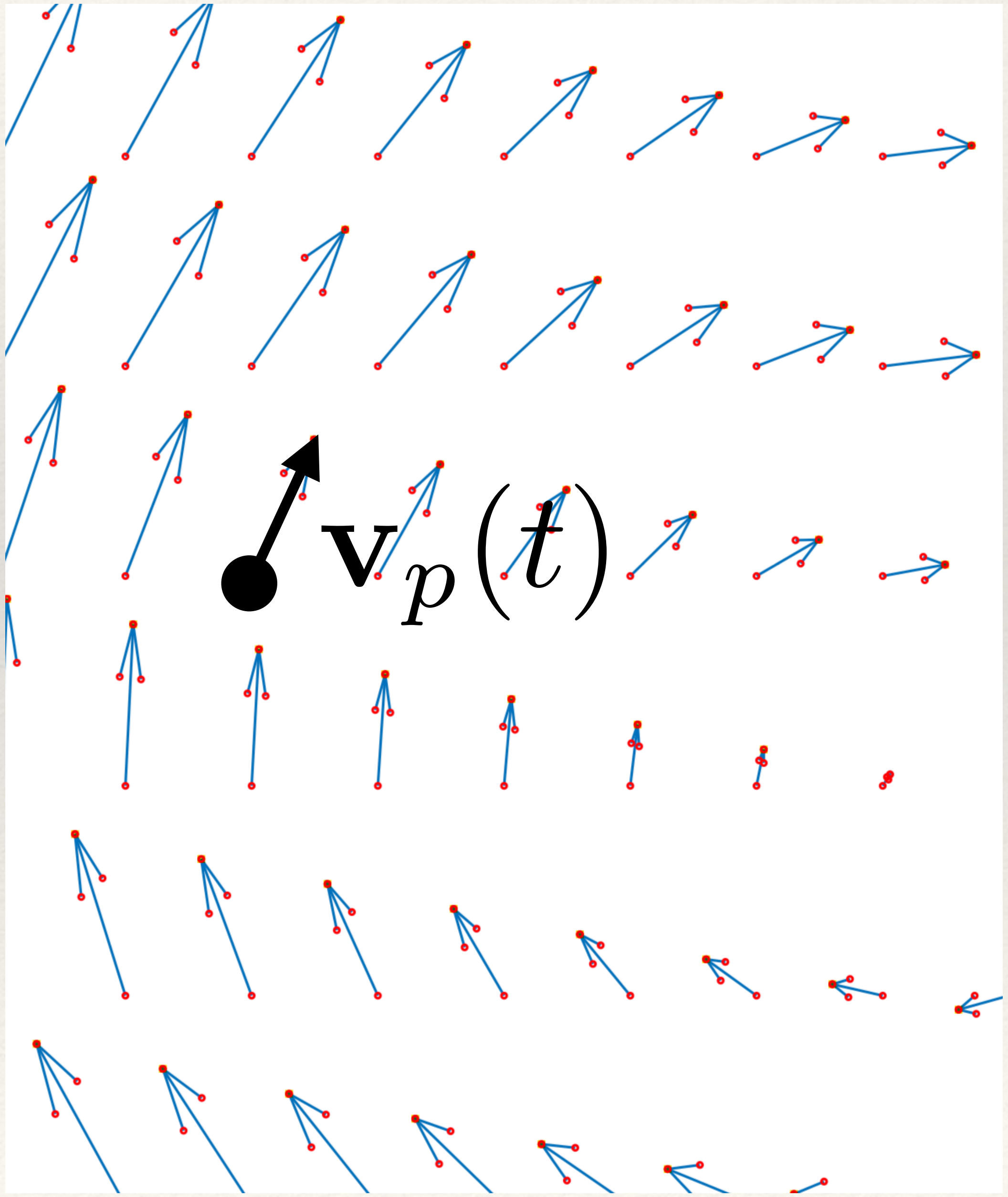
$$\mathbf{a}_p(t) = \frac{d}{dt}\mathbf{v}_p(t)$$
$$= \frac{d}{dt}\mathbf{u}(\mathbf{x}_p(t), t)$$



Material Derivative

$$\mathbf{u}(\mathbf{x}, t)$$

$$\begin{aligned}\mathbf{a}_p(t) &= \frac{d}{dt}\mathbf{v}_p(t) \\ &= \frac{d}{dt}\mathbf{u}(\mathbf{x}_p(t), t) \\ &= \left(\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\frac{d\mathbf{x}_p}{dt}\right)\end{aligned}$$

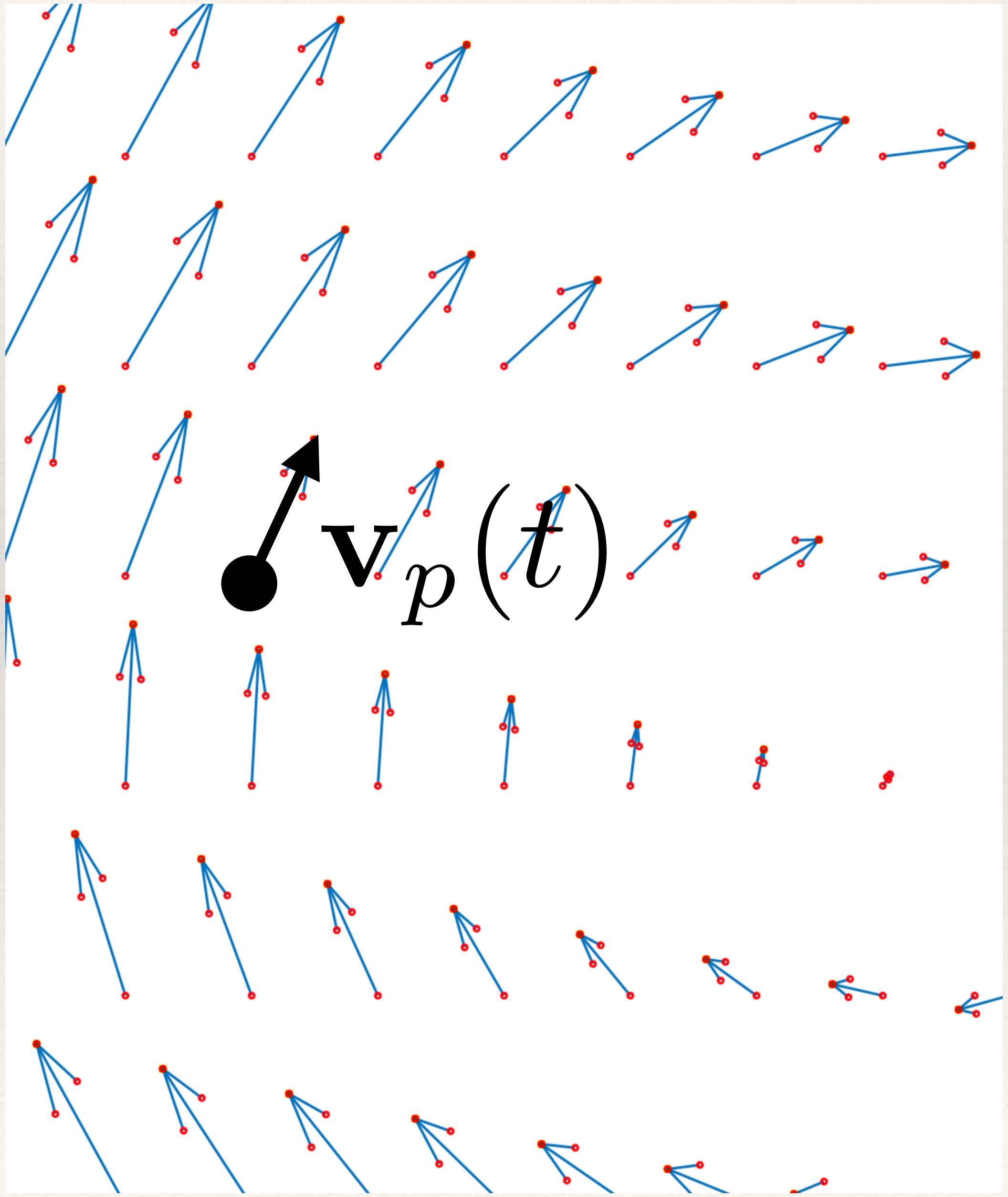


Material Derivative

$$\mathbf{u}(\mathbf{x}, t)$$

$$\mathbf{a}_p(t) = \frac{d}{dt}\mathbf{v}_p(t)$$
$$= \frac{d}{dt}\mathbf{u}(\mathbf{x}_p(t), t)$$
$$= \underbrace{\left(\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\frac{d\mathbf{x}_p}{dt}\right)}_{\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}}$$

$$\frac{D\mathbf{u}}{Dt} = \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}$$



Pressure Forces

$$\overset{\text{m}}{\rho}\,\overbrace{(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u})}^{\mathbf{a}} = \overbrace{-\nabla p + \mu\Delta\mathbf{u} + \mathbf{f}}^{\mathbf{f}}$$

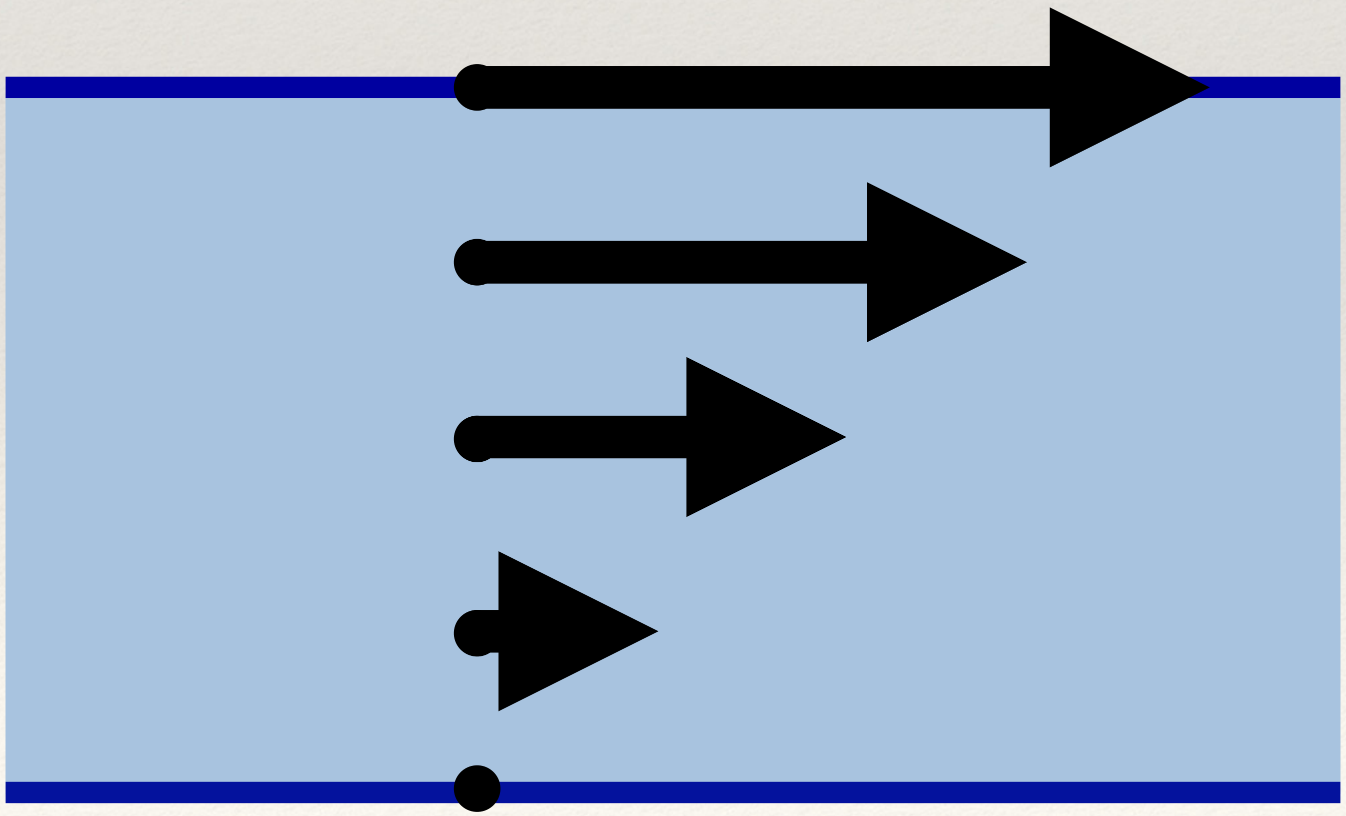
$$\nabla\cdot\mathbf{u} = 0$$

- maintains fluid volume, resisting compression/expansion
- forces fluid from areas of high pressure to low pressure

Viscous Forces

$$\overbrace{\rho}^{\mathrm{m}}\,\overbrace{\left(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}\right)}^{\mathbf{a}} = \overbrace{-\nabla p + \mu\Delta\mathbf{u} + \mathbf{f}}^{\mathbf{f}}$$

$$\nabla\cdot\mathbf{u} = 0$$



Viscous Forces

$$\overbrace{\rho}^{\text{m}}\overbrace{\left(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}\right)}^{\mathbf{a}} = -\overbrace{\nabla p + \mu\Delta\mathbf{u} + \mathbf{f}}^{\mathbf{f}}$$

$$\nabla\cdot\mathbf{u} = 0$$

- internal friction
- Laplacian measures difference from neighbors

μ	$(\mathrm{mPa}\cdot\mathrm{s})$
μ_{air}	1.8×10^{-2}
μ_{water}	1
μ_{honey}	5×10^{3}

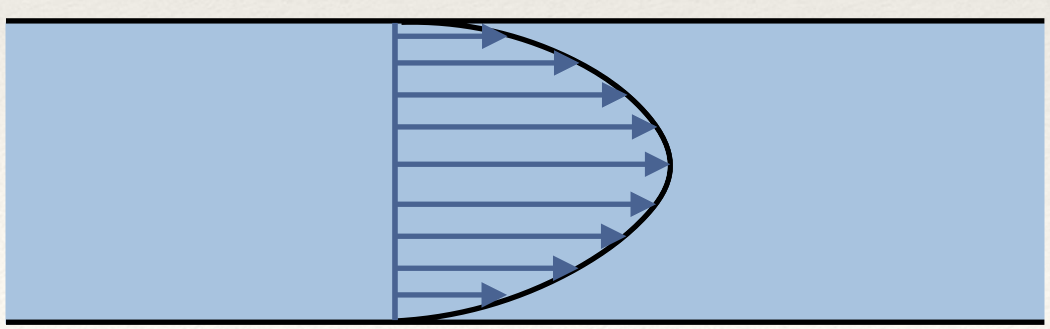
Viscous Forces: Solid Boundaries

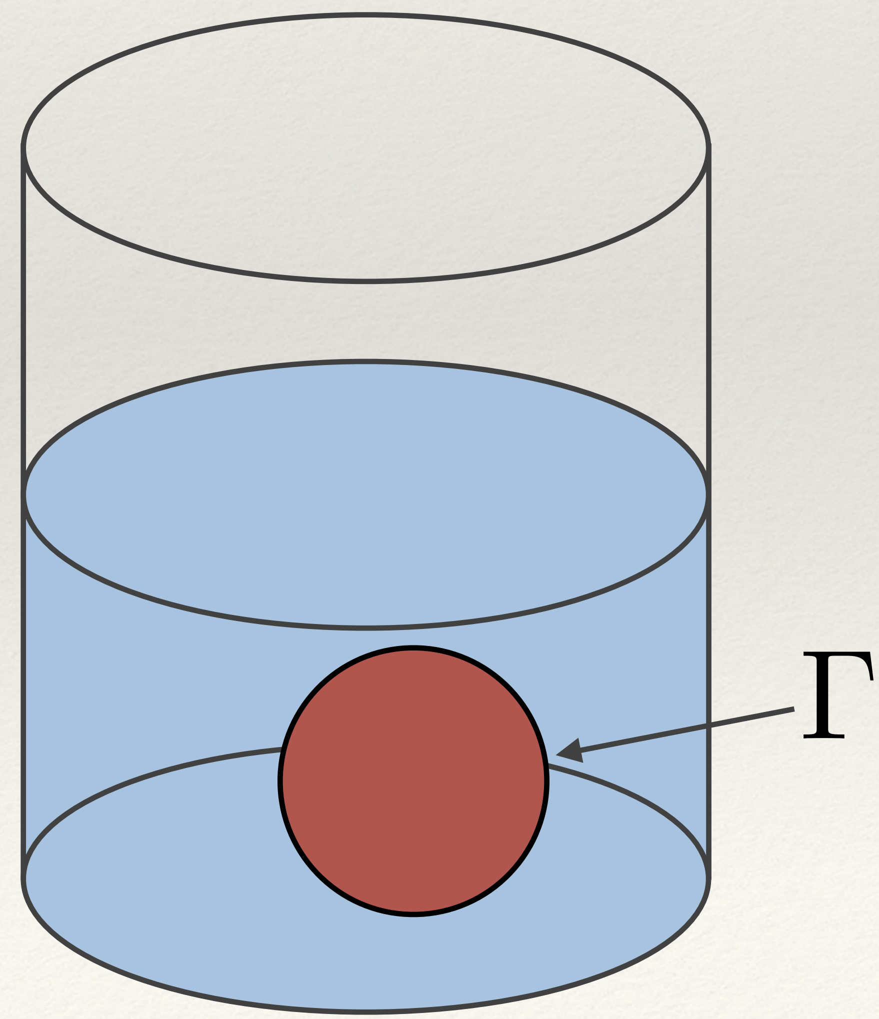
$$\overbrace{\rho}^{\text{m}} \overbrace{(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u})}^{\mathbf{a}} = \overbrace{-\nabla p + \mu\Delta\mathbf{u} + \mathbf{f}}^{\mathbf{f}}$$

$$\nabla\cdot\mathbf{u} = 0$$

- no-slip boundary condition

$$\mathbf{u}(\mathbf{x},t) = \mathbf{V}(\mathbf{x},t), \quad \mathbf{x}\in\Gamma$$





External Forces

$$\overbrace{\rho}^{\text{m}}\,\overbrace{\left(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}\right)}^{\mathbf{a}} = \overbrace{-\nabla p + \mu \Delta \mathbf{u} + \mathbf{f}}^{\mathbf{f}}$$

$$\nabla \cdot \mathbf{u} = 0$$

- Gravity, surface tension, interaction forces, control forces, embedded structures

Incompressibility

$$\overset{\text{m}}{\rho}\,\overbrace{\left(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}\right)}^{\mathbf{a}} = -\overbrace{\nabla p + \mu\Delta\mathbf{u} + \mathbf{f}}^{\mathbf{f}}$$

$$\boxed{\nabla\cdot\mathbf{u} = 0}$$

- Conservation of mass $\qquad \dfrac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0$
- Constant density $\qquad \Rightarrow \nabla\cdot\mathbf{u} = 0$

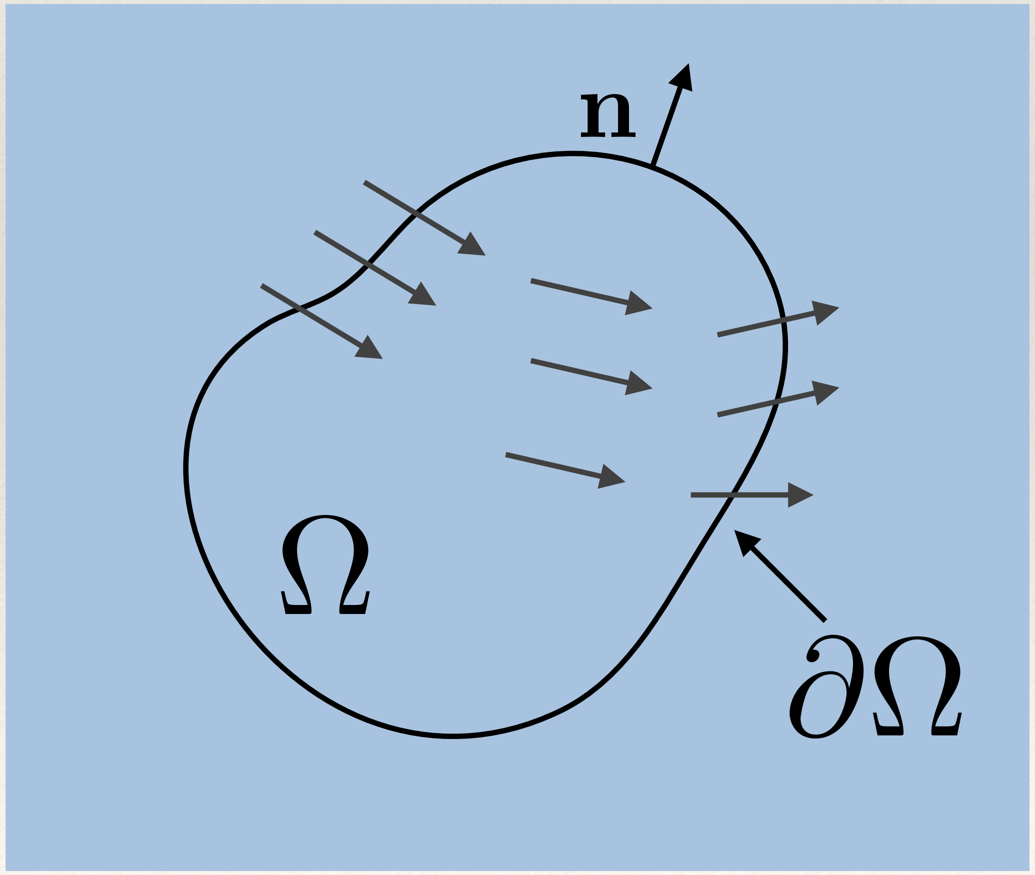
Incompressibility

$$\overbrace{\rho}^{\text{m}}\overbrace{(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u})}^{\mathbf{a}} = \overbrace{-\nabla p + \mu\Delta\mathbf{u} + \mathbf{f}}^{\mathbf{f}}$$

$$\nabla \cdot \mathbf{u} = 0$$

- Net flow through boundary must be zero

$$0 = \int_\Omega \nabla \cdot \mathbf{u} = \int_{\partial\Omega} \mathbf{u}\cdot\mathbf{n}$$



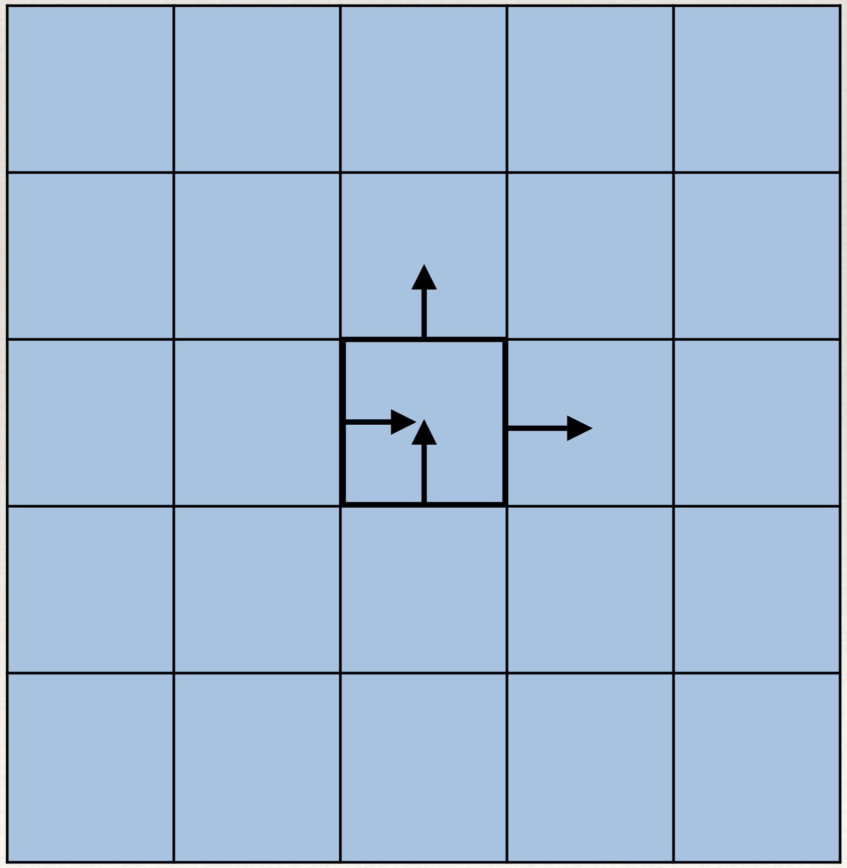
Incompressibility

$$\overset{\text{m}}{\rho}\,\overbrace{(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u})}^{\mathbf{a}} = \overbrace{-\nabla p + \mu\Delta\mathbf{u} + \mathbf{f}}^{\mathbf{f}}$$

$$\nabla\cdot\mathbf{u} = 0$$

- Net flow through boundary must be zero

$$0 = \int_\Omega \nabla\cdot\mathbf{u} = \int_{\partial\Omega}\mathbf{u}\cdot\mathbf{n}$$



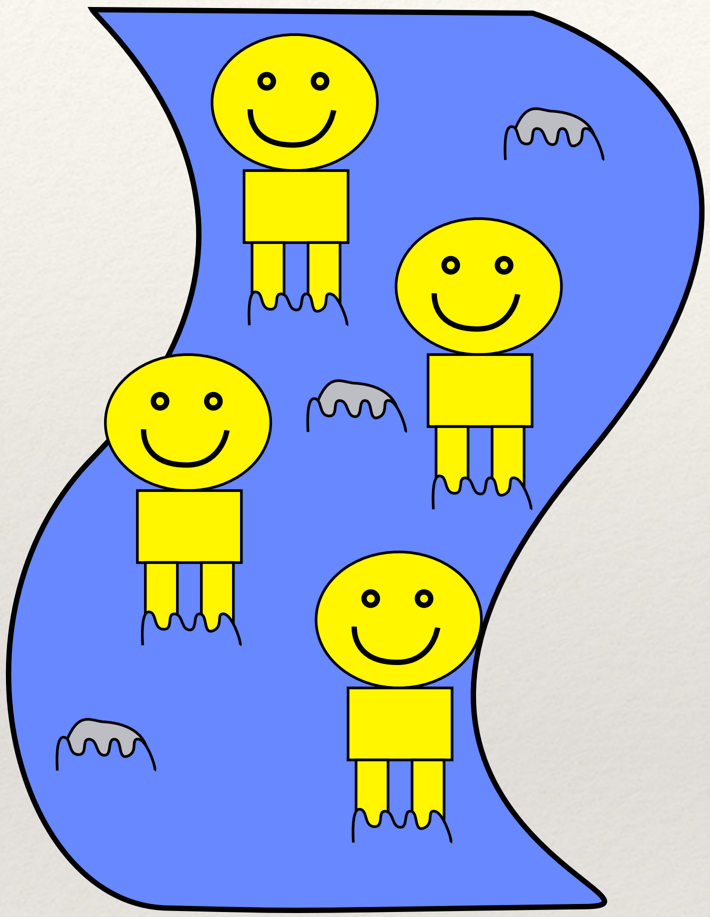
III. Spatial Discretization

Lagrangian vs. Eulerian

Reference Frames

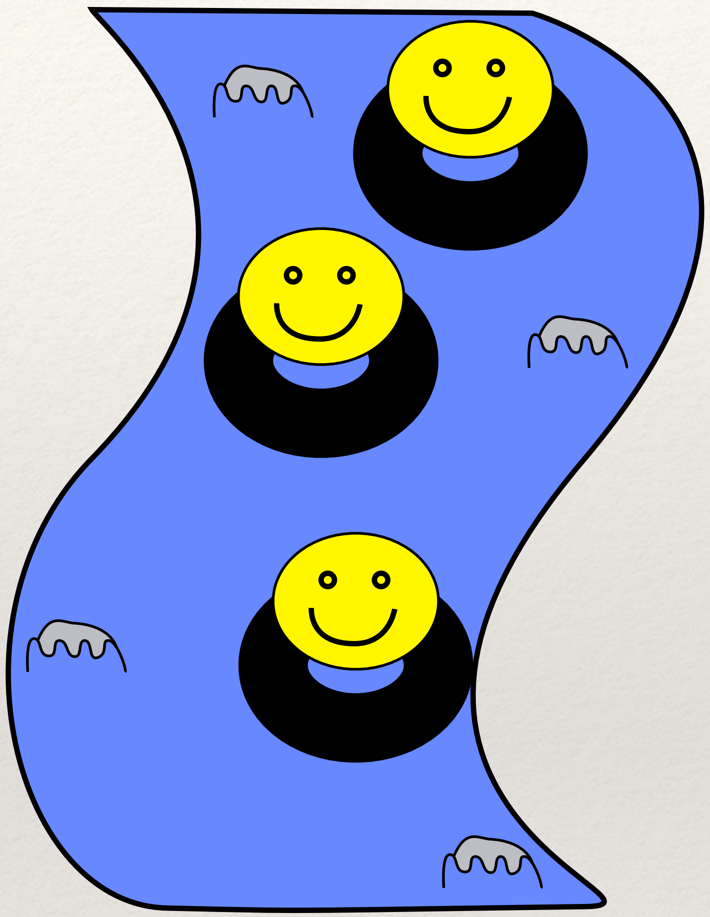
- An Eulerian reference frame is fixed.
- A Lagrangian reference frame moves with the material.

Reference Frames



Eulerian

$$\frac{\partial y}{\partial t} + \mathbf{u} \cdot \nabla y$$



Lagrangian

$$\frac{dy}{dt}$$

Grids, Meshes, and Particles

Regular Grids

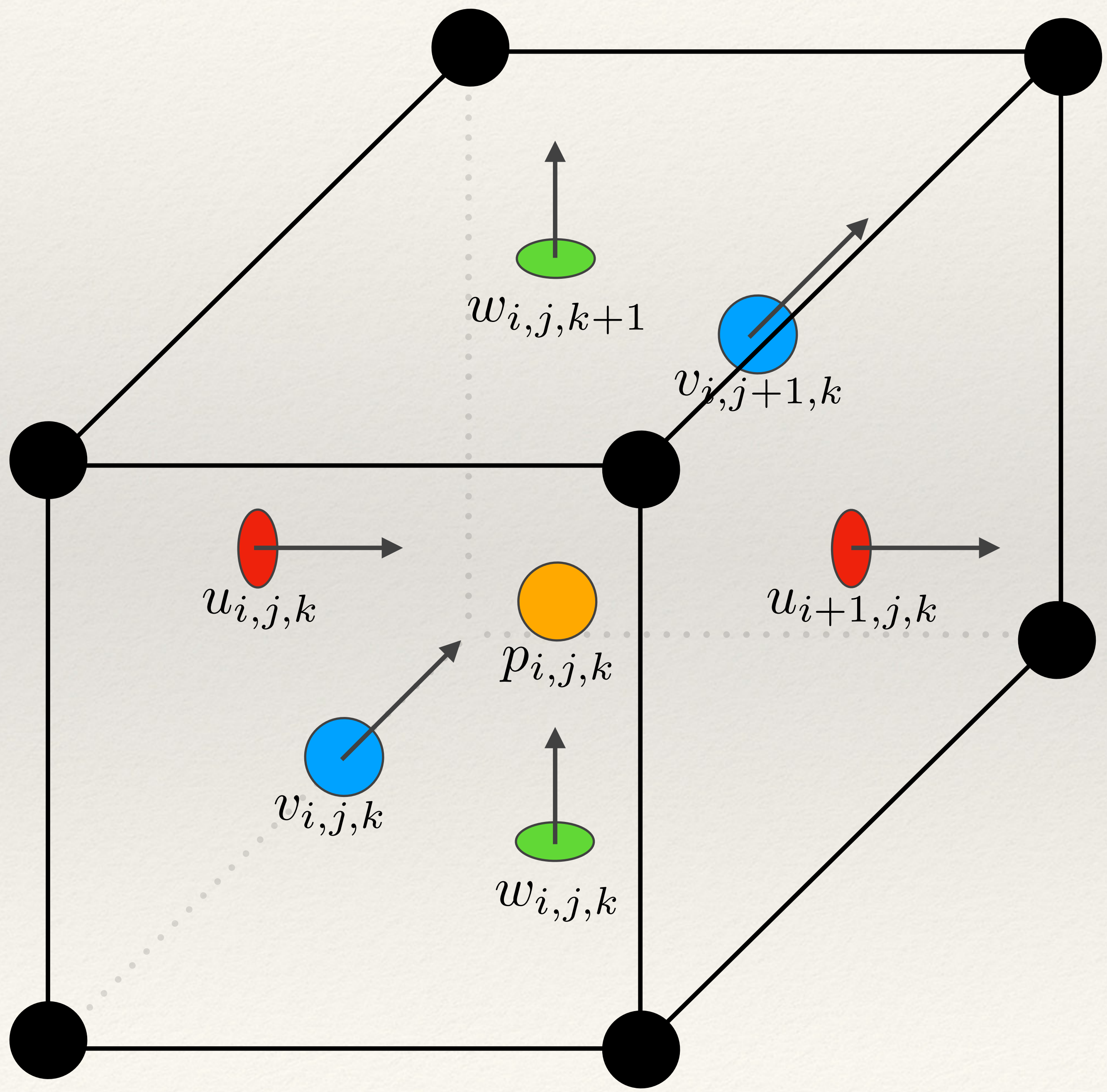
Advantages

- Simple
- Fast operations (e.g. point location)
- Can take advantage of structure for efficiency

Disadvantages

- Difficult to track shape over time
- Difficult to handle non-grid-aligned boundaries

Staggered Grid



Meshes (Simplicial Complexes)

Advantages

- Easy to map to previous points in time
- Can conform to boundaries

Disadvantages

- Difficult to generate meshes
- Difficult to perform some operations (e.g. point location)

Particles

Advantages

- Simple
- Easy to map to previous points in time

Disadvantages

- Difficult to perform integration because they don’t partition space

Hybrid Structures

Advantages

- Advantages of underlying structures

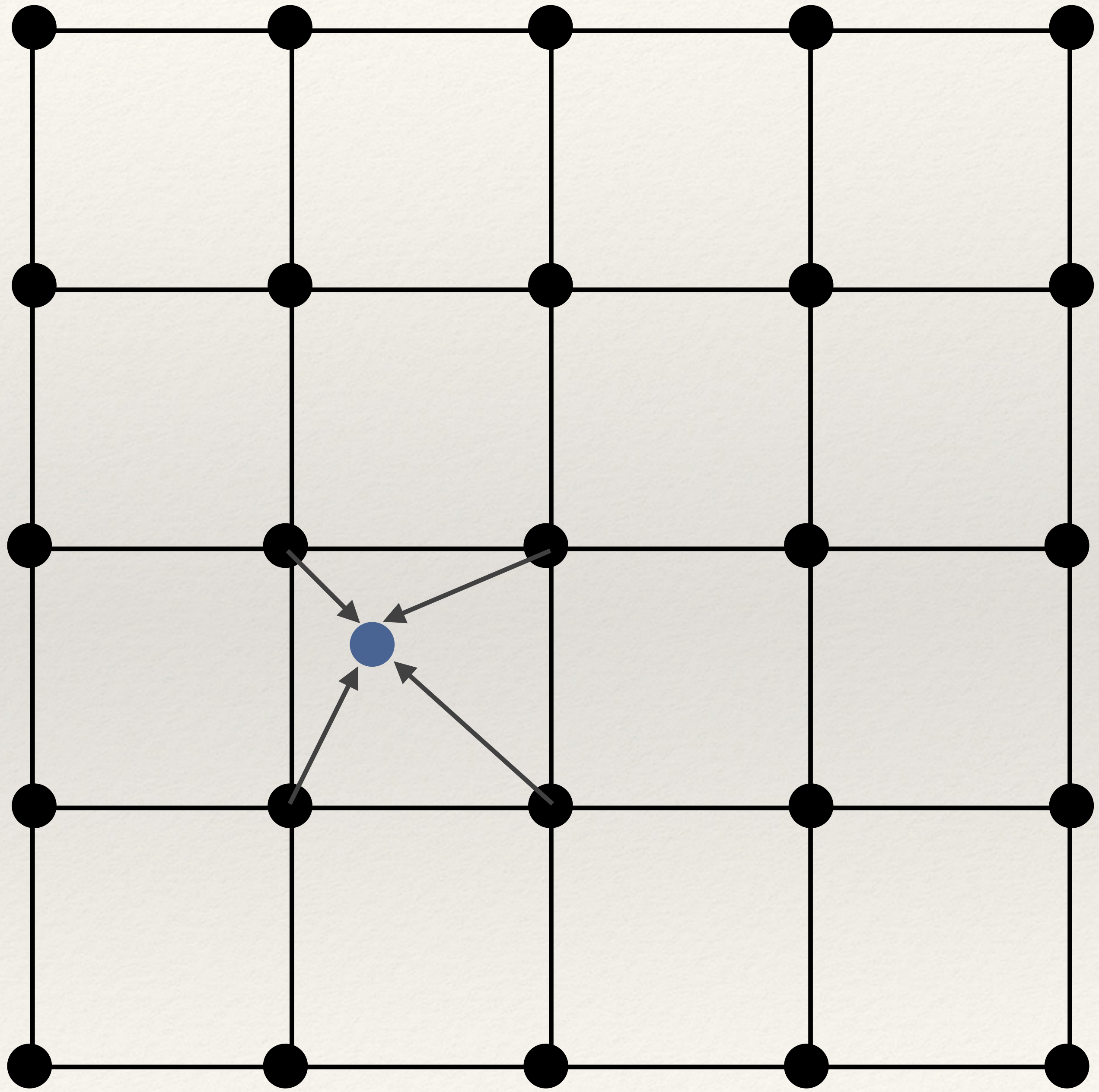
Disadvantages

- Complexity
- Computational and accuracy costs from mapping between structures

Interpolation

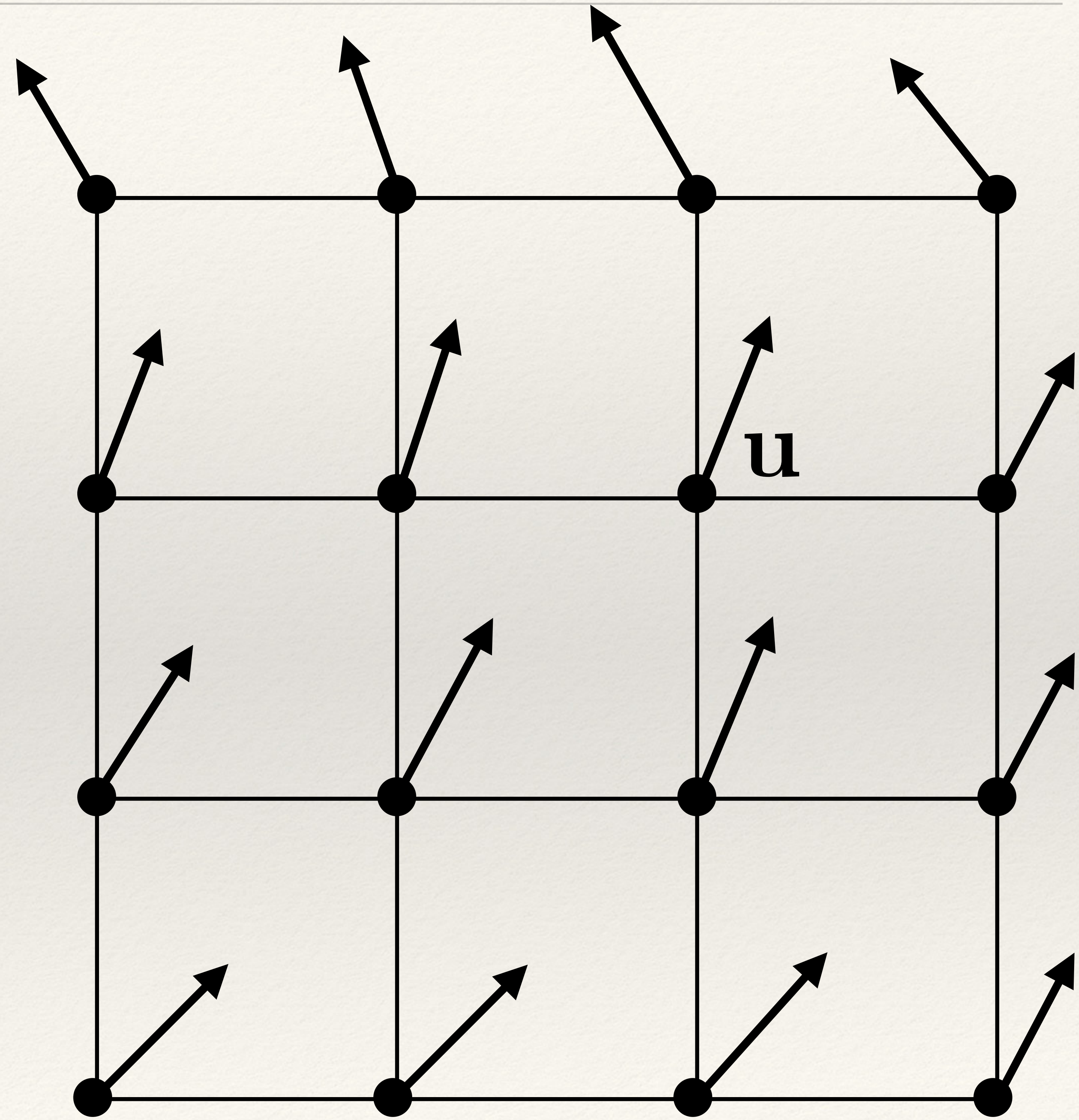
Interpolation

- Samples stored at discrete points on grid, mesh, or particles
- Elsewhere, must be interpolated there



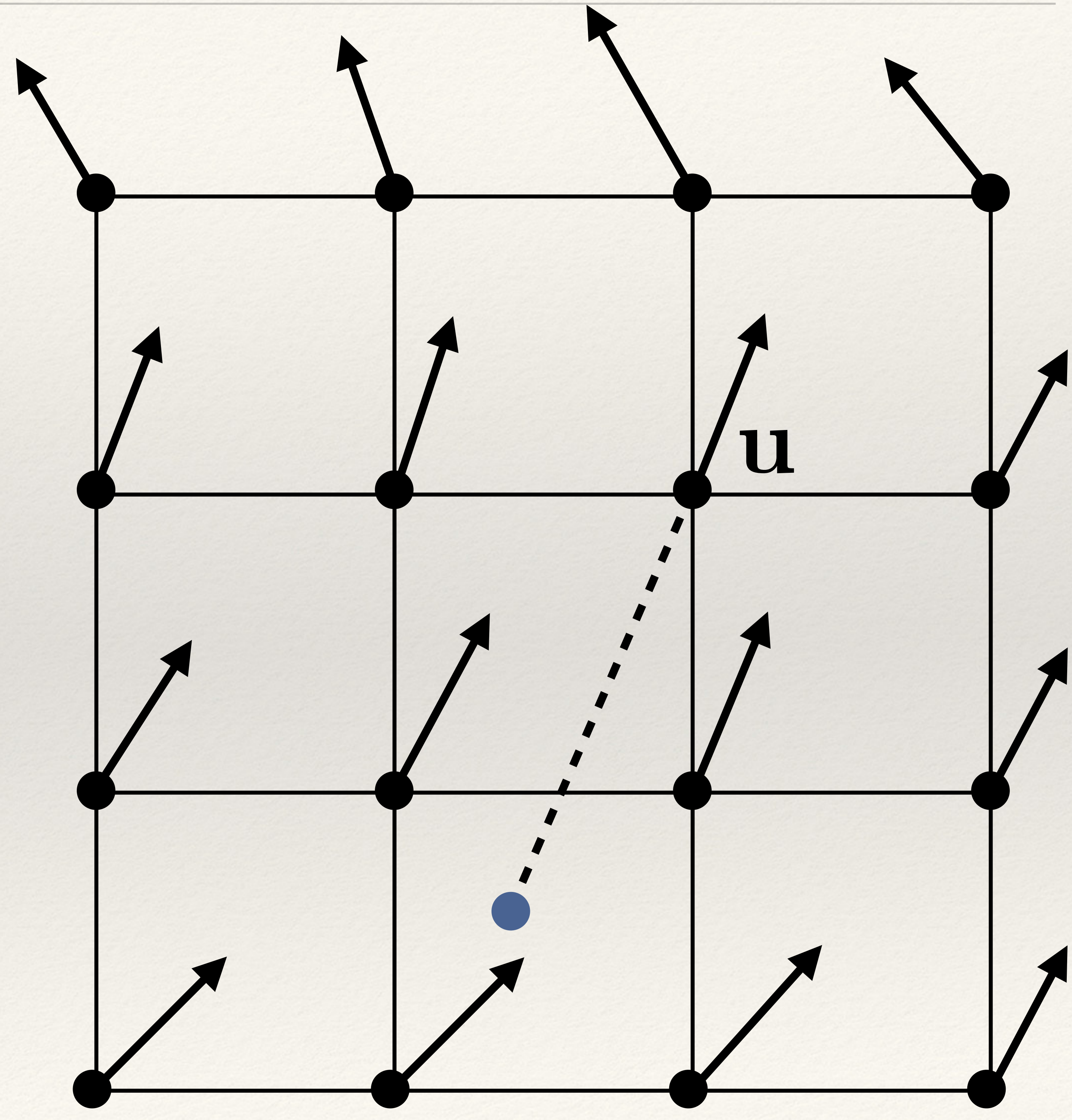
Interpolation

- Example: semi-Lagrangian advection



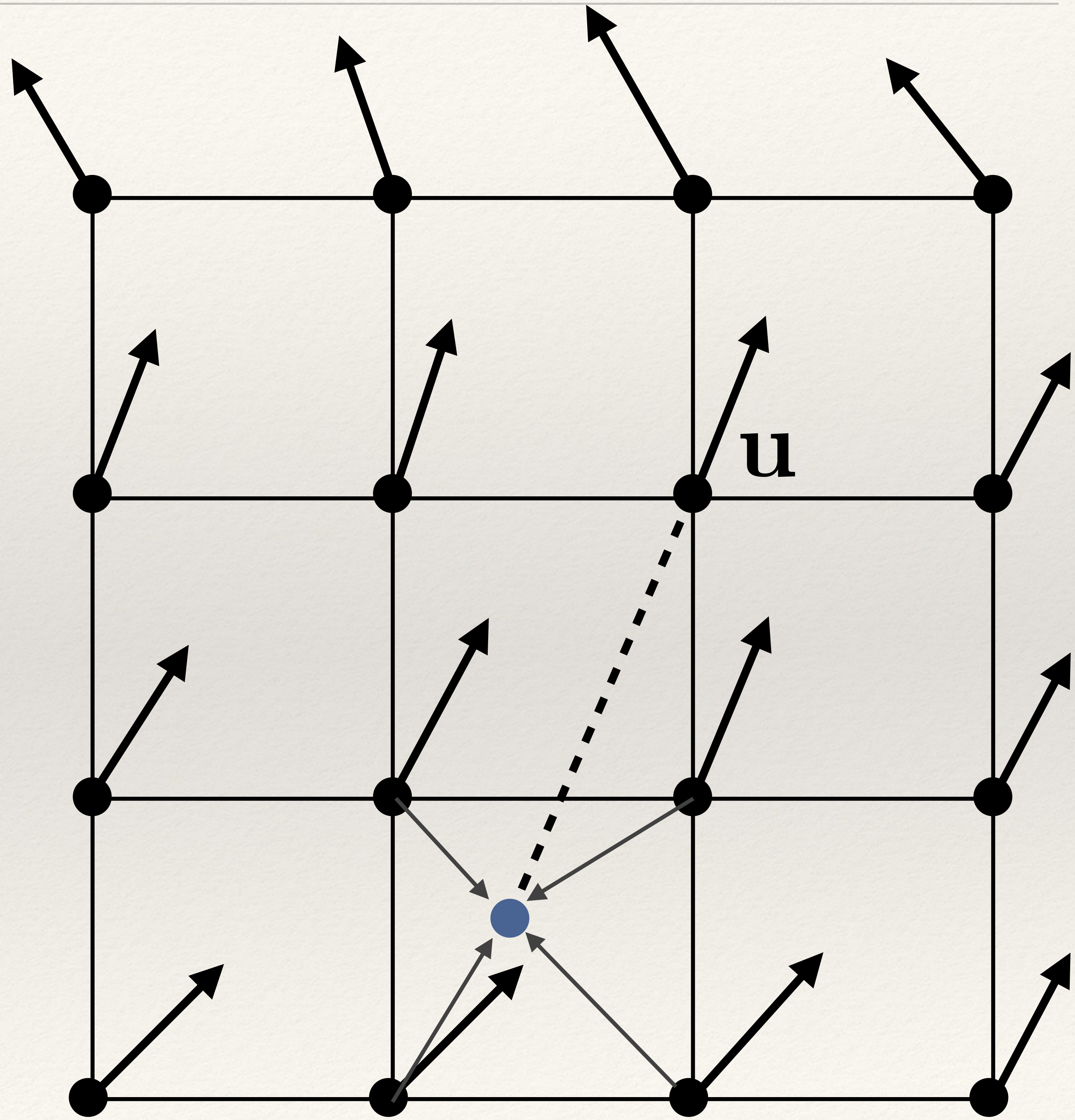
Interpolation

- Example: semi-Lagrangian advection



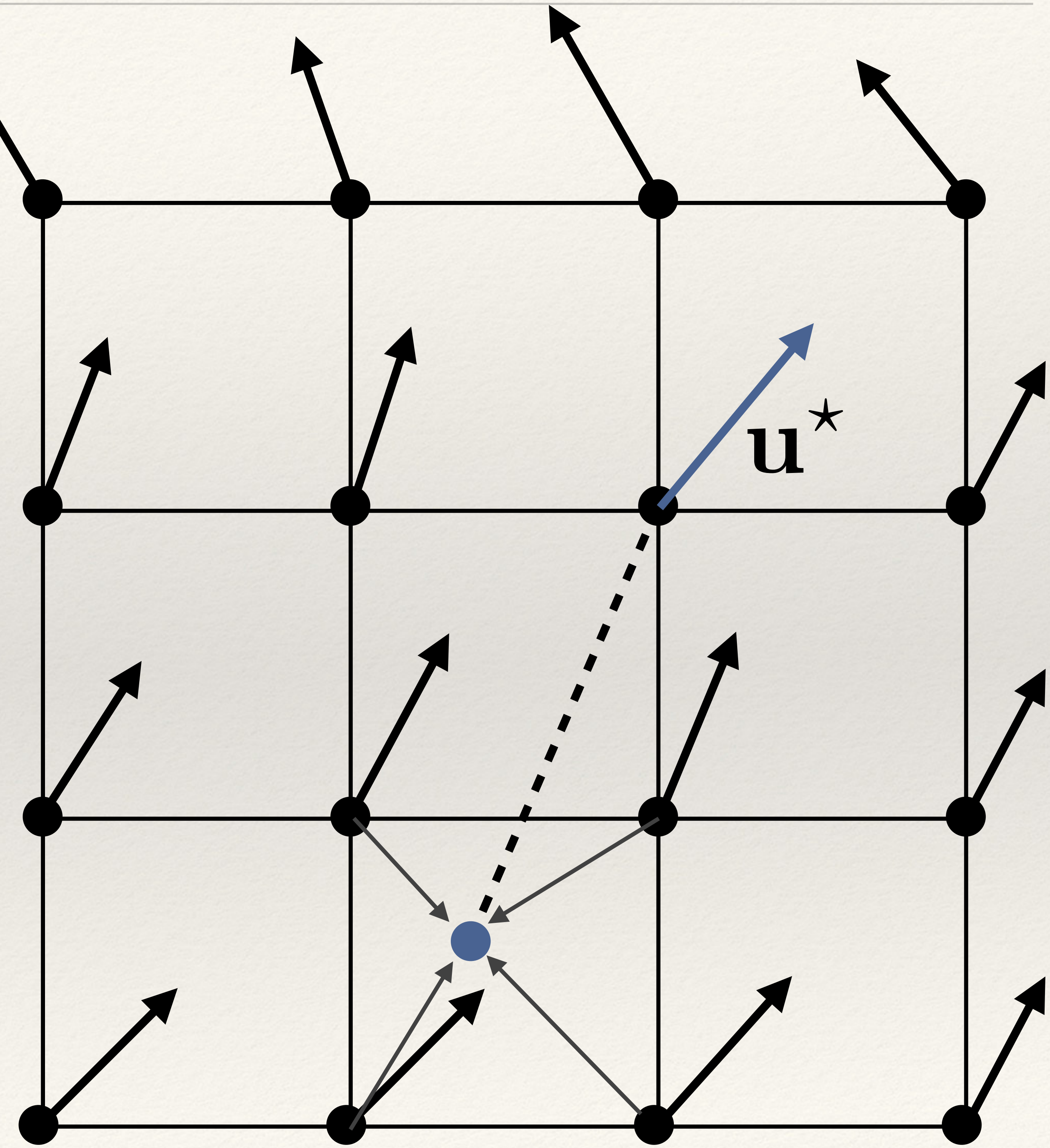
Interpolation

- Example: semi-Lagrangian advection



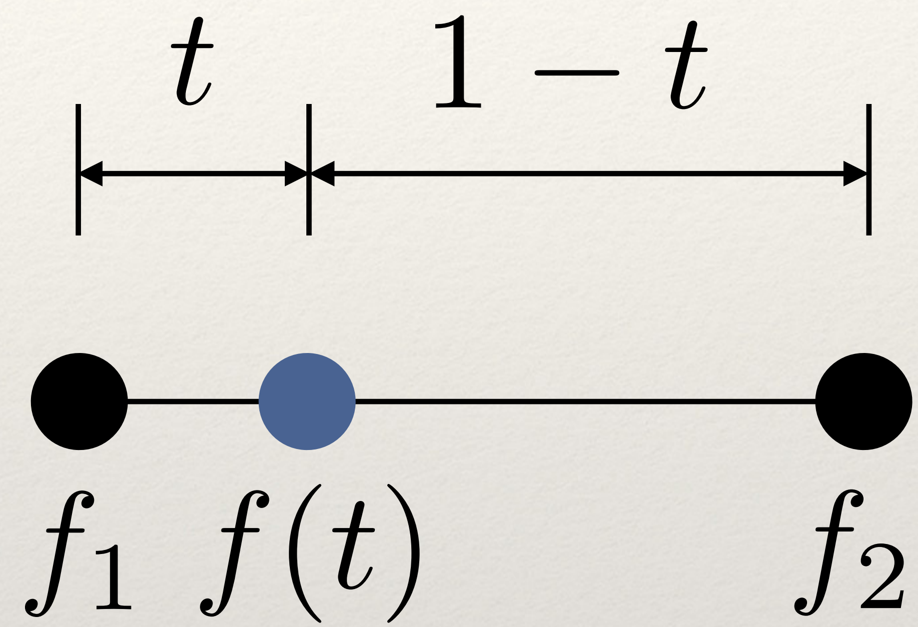
Interpolation

- Example: semi-Lagrangian advection



Linear Interpolation (1D)

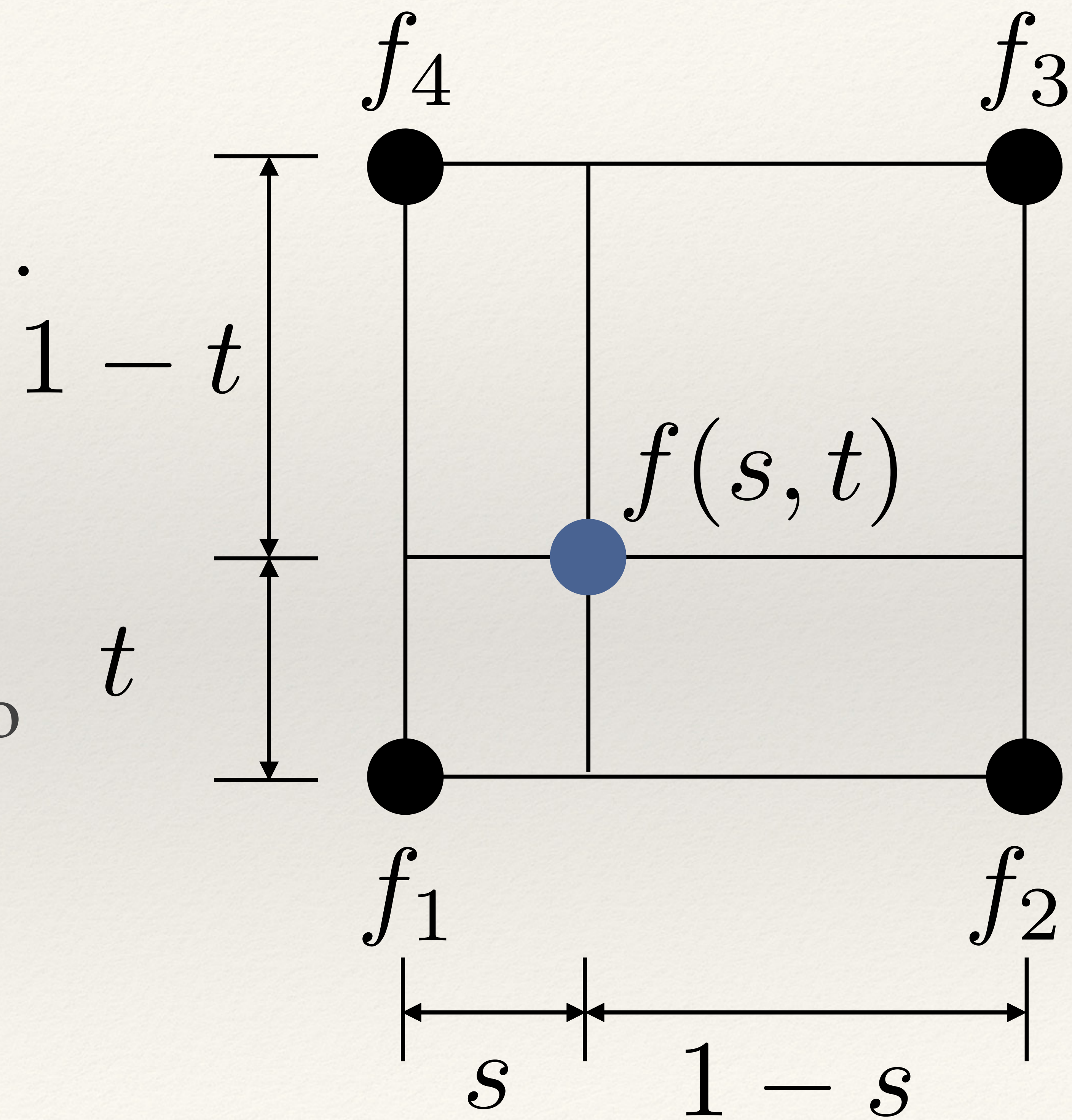
- $f(0) = f_1$
- $f(1) = f_2$
- weights sum to 1
- weight is length ratio opposite point



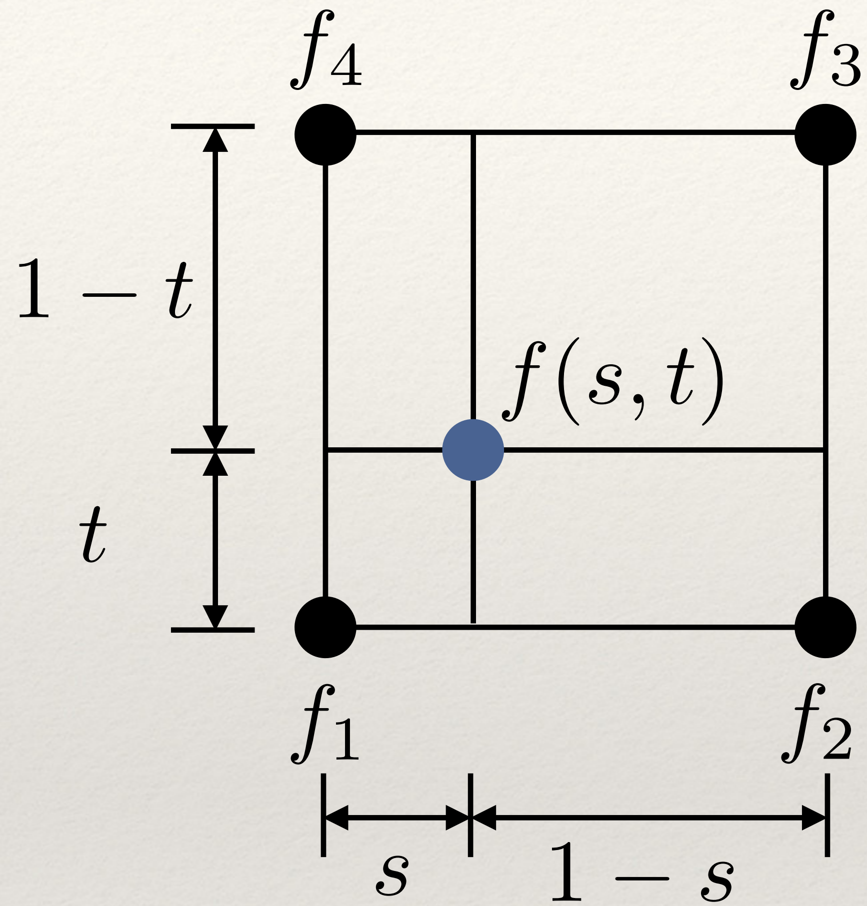
$$f(t) = (1-t)f_1 + tf_2.$$

Bilinear Interpolation (2D)

- $f(0,0) = f_1, \cdots$
- $f(0,1) = f_4$
- weights sum to 1
- weight is area ratio opposite point

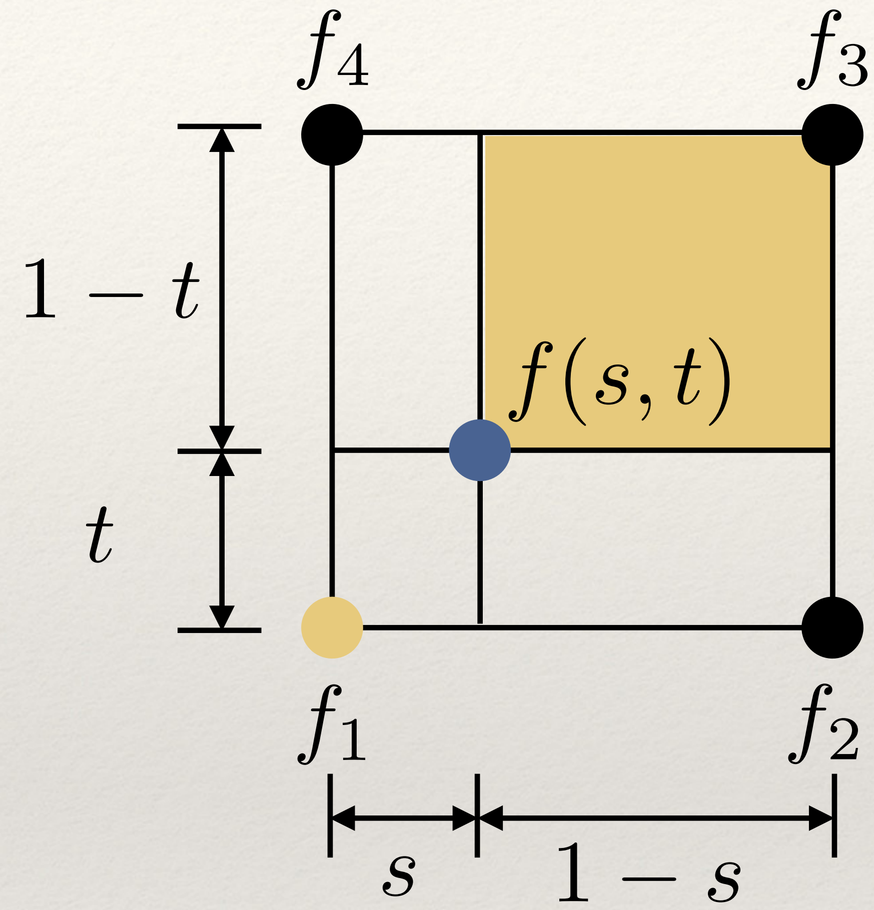


Bilinear Interpolation (2D)



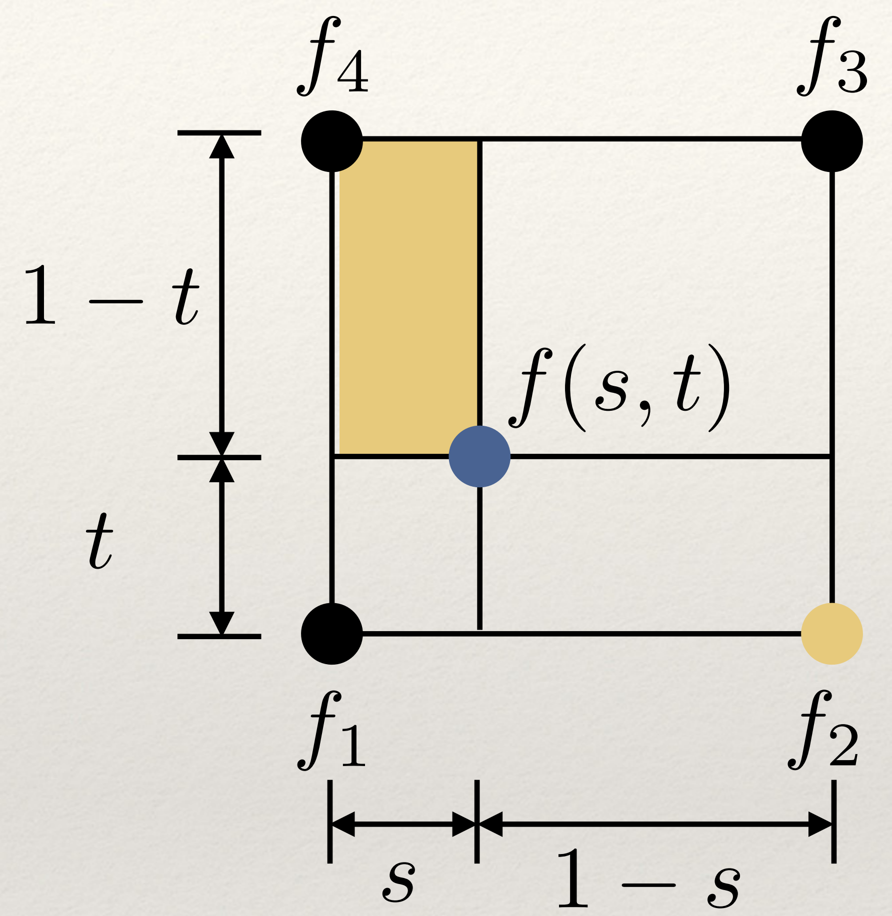
$$f(s,t) = (1-s)(1-t)f_1 + s(1-t)f_2 + stf_3 + (1-s)tf_4$$

Bilinear Interpolation (2D)



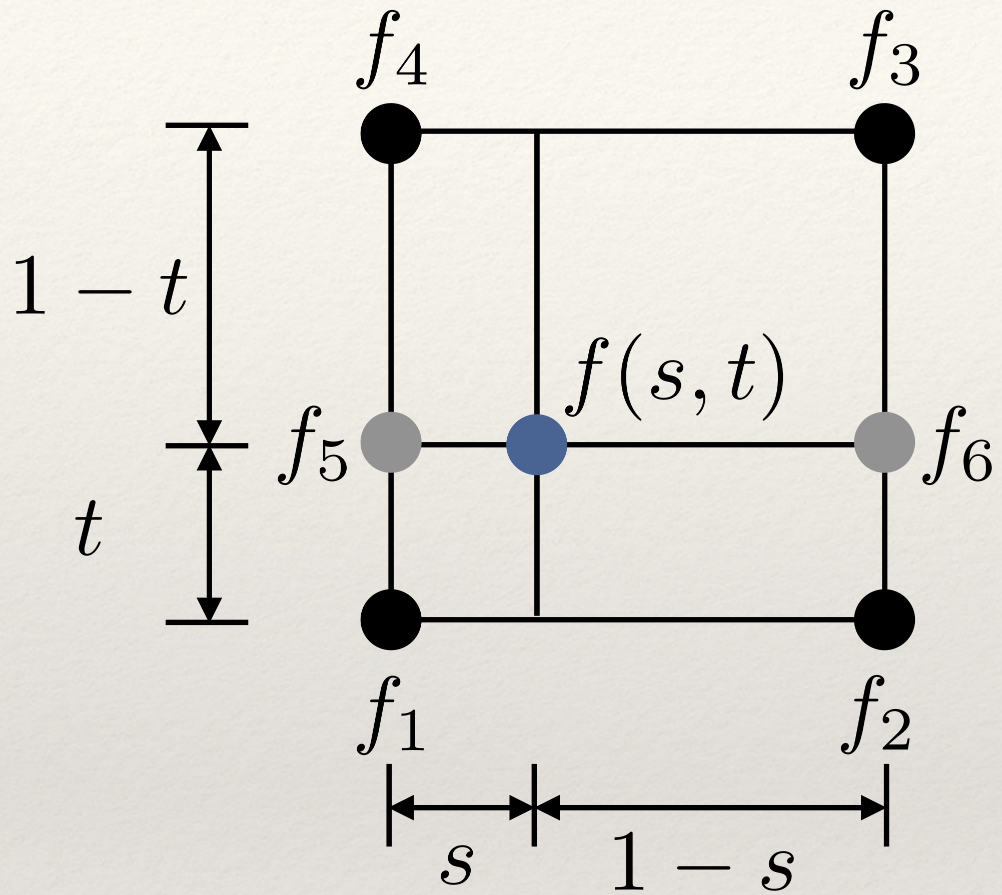
$$f(s,t) = (1-s)(1-t)f_1 + s(1-t)f_2 + stf_3 + (1-s)tf_4$$

Bilinear Interpolation (2D)



$$f(s,t) = (1-s)(1-t)f_1 + s(1-t)f_2 + stf_3 + (1-s)tf_4$$

Bilinear Interpolation (2D)

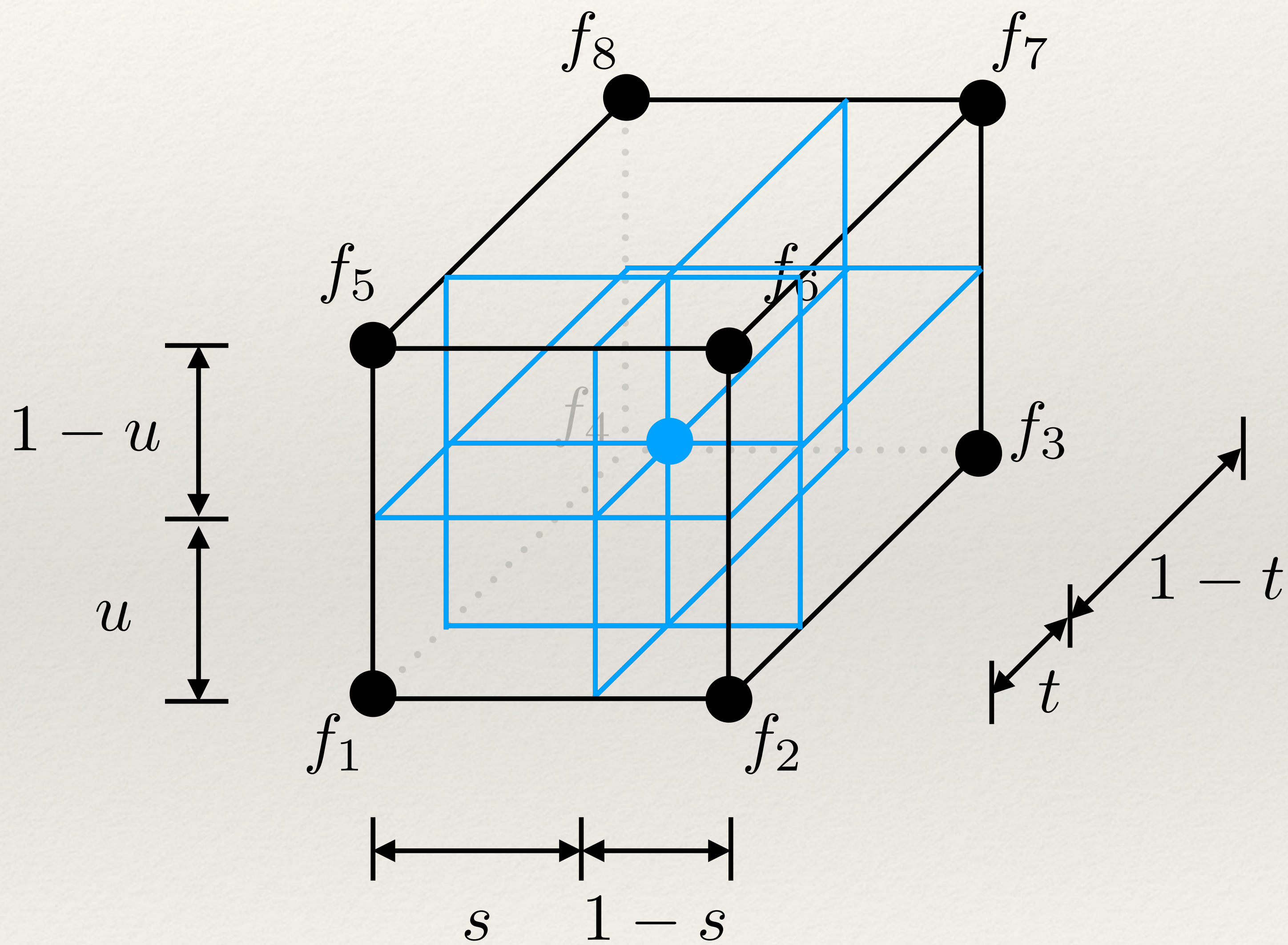


$$f_5 = (1-t)f_1 + tf_4 \qquad f_6 = (1-t)f_2 + tf_3$$

$$f(s,t) = (1-s)(1-t)f_1 + s(1-t)f_2 + stf_3 + (1-s)tf_4$$

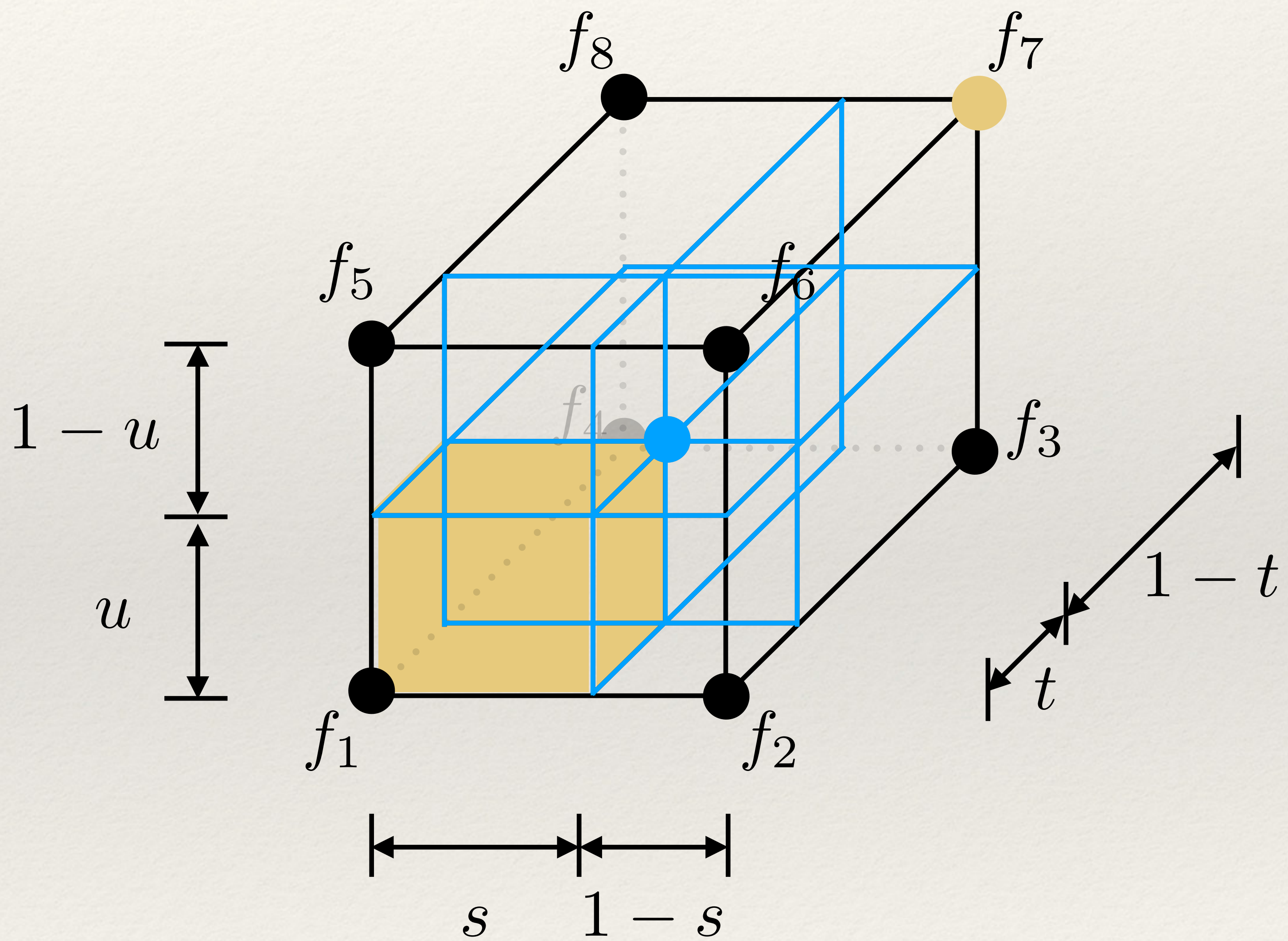
Trilinear Interpolation (3D)

$$
\begin{aligned}
f(s,t,u) &= \\
&(1-s)(1-t)(1-u)f_1 \\
&+ s(1-t)(1-u)f_2 \\
&+ st(1-u)f_3 \\
&+ (1-s)t(1-u)f_4 \\
&+ (1-s)(1-t)uf_5 \\
&+ s(1-t)uf_6 \\
&+ stuf_7 \\
&+ (1-s)tuf_8
\end{aligned}
$$



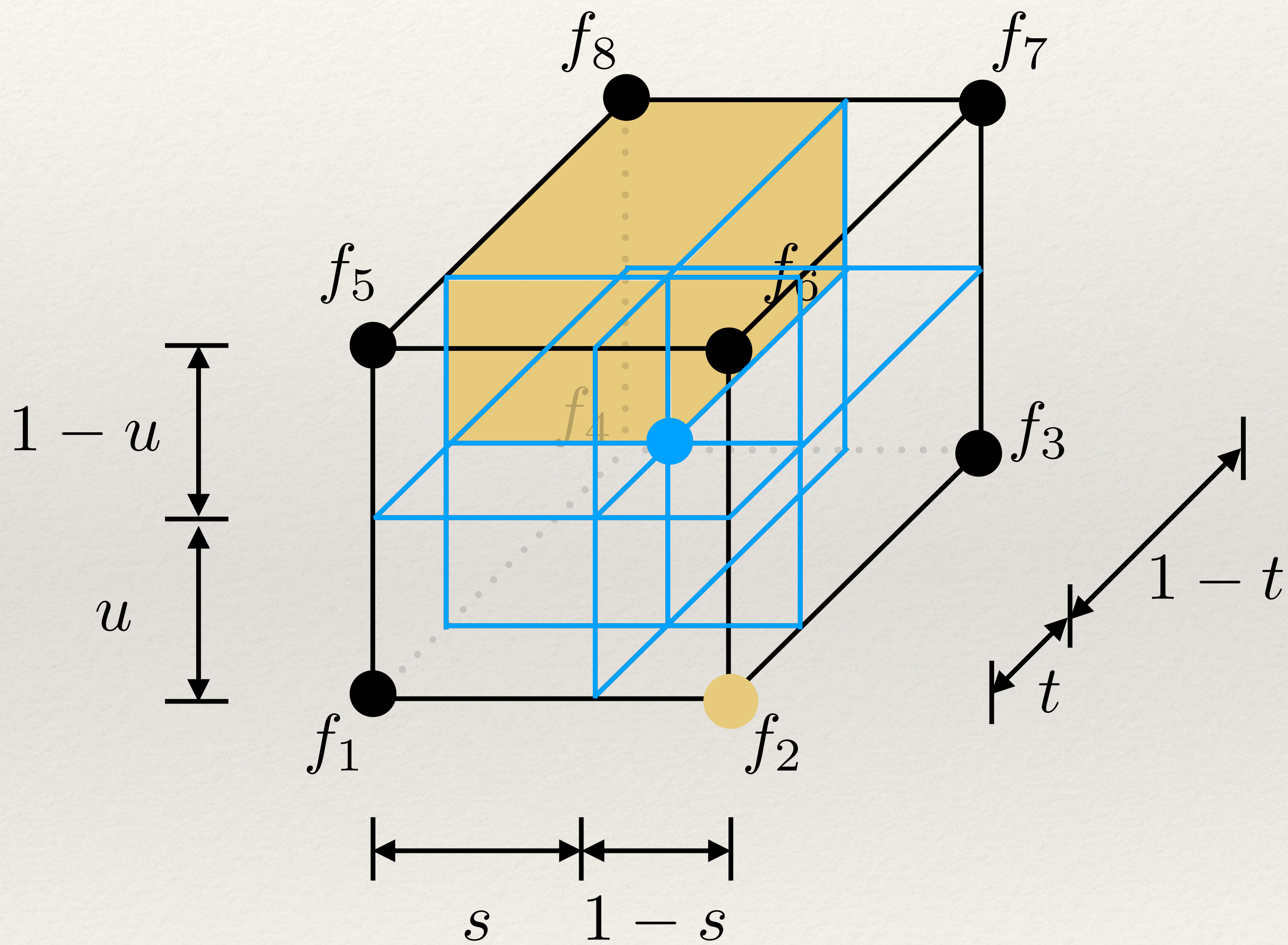
Trilinear Interpolation (3D)

$$f(s,t,u) =$$
$$(1-s)(1-t)(1-u)f_1$$
$$+ s(1-t)(1-u)f_2$$
$$+ st(1-u)f_3$$
$$+ (1-s)t(1-u)f_4$$
$$+ (1-s)(1-t)uf_5$$
$$+ s(1-t)uf_6$$
$$+ stuf_7$$
$$+ (1-s)tuf_8$$



Trilinear Interpolation (3D)

$$f(s,t,u) =$$
$$(1-s)(1-t)(1-u)f_1$$
$$+ s(1-t)(1-u)f_2$$
$$+ st(1-u)f_3$$
$$+ (1-s)t(1-u)f_4$$
$$+ (1-s)(1-t)uf_5$$
$$+ s(1-t)uf_6$$
$$+ stuf_7$$
$$+ (1-s)tuf_8$$



Trilinear Interpolation (3D)

$f(s,t,u) =$

$(1-s)(1-$

$+\ s(1-t)($

$+\ st(1-u)$

$+\ (1-s)t($

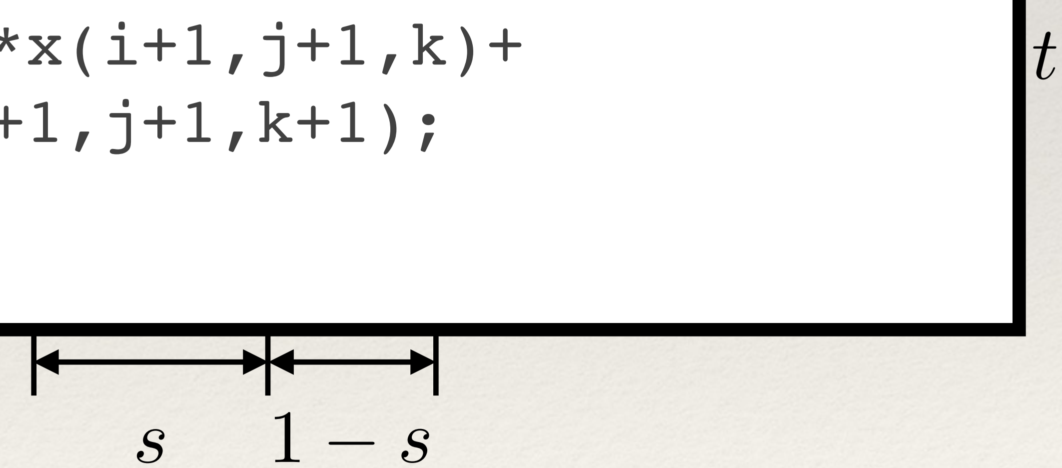
$+\ (1-s)(1$

$+\ s(1-t)u$

$+\ stuf_7$

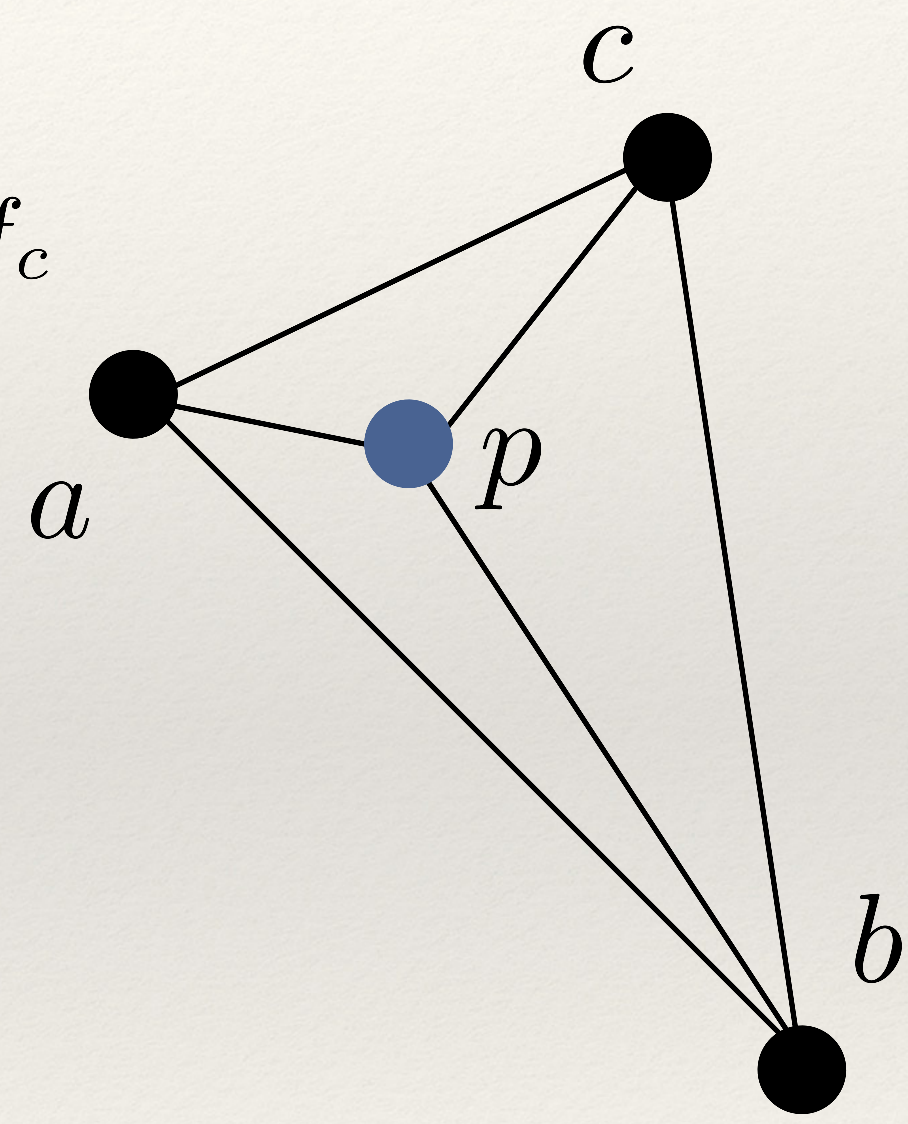
$+\ (1-s)tuf_8$

```
inline double trilinearInterp(const
Array3D<double> &x, int i, int j, int k,
    double w0, double w1, double w2) {
  return (1.0-w0)*(1.0-w1)*(1.0-w2)*x(i,j,k)+
    (1.0-w0)*(1.0-w1)*(w2)*x(i,j,k+1)+
    (1.0-w0)*(w1)*(1.0-w2)*x(i,j+1,k)+
    (1.0-w0)*(w1)*(w2)*x(i,j+1,k+1)+
    (w0)*(1.0-w1)*(1.0-w2)*x(i+1,j,k)+
    (w0)*(1.0-w1)*(w2)*x(i+1,j,k+1)+
    (w0)*(w1)*(1.0-w2)*x(i+1,j+1,k)+
    (w0)*(w1)*(w2)*x(i+1,j+1,k+1);
}
```



Barycentric Coordinates

$$f_p = f(\alpha, \beta, \gamma) = \alpha f_a + \beta f_b + \gamma f_c$$



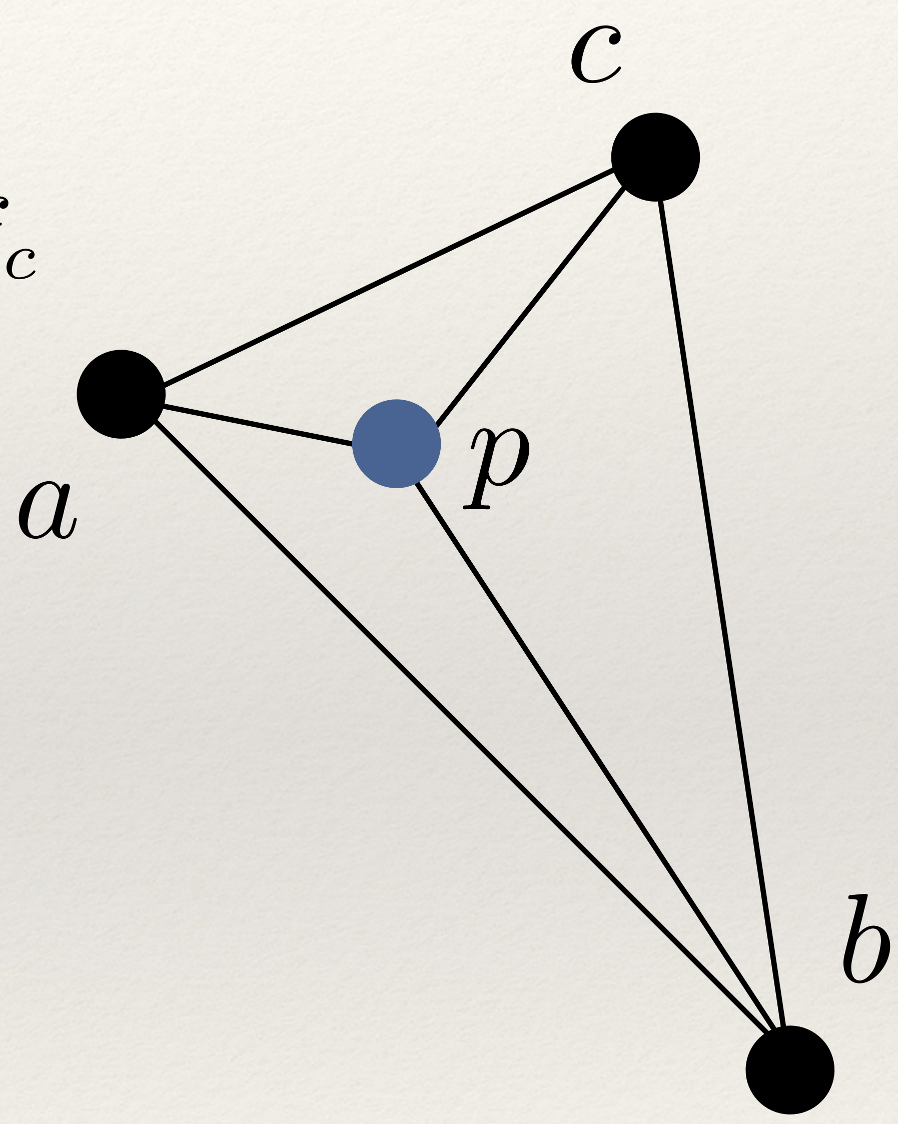
Barycentric Coordinates

$$f_p = f(\alpha, \beta, \gamma) = \alpha f_a + \beta f_b + \gamma f_c$$

$$\alpha = \frac{\text{area}(p, b, c)}{\text{area}(a, b, c)},$$

$$\beta = \frac{\text{area}(p, c, a)}{\text{area}(a, b, c)},$$

$$\gamma = \frac{\text{area}(p, a, b)}{\text{area}(a, b, c)}.$$



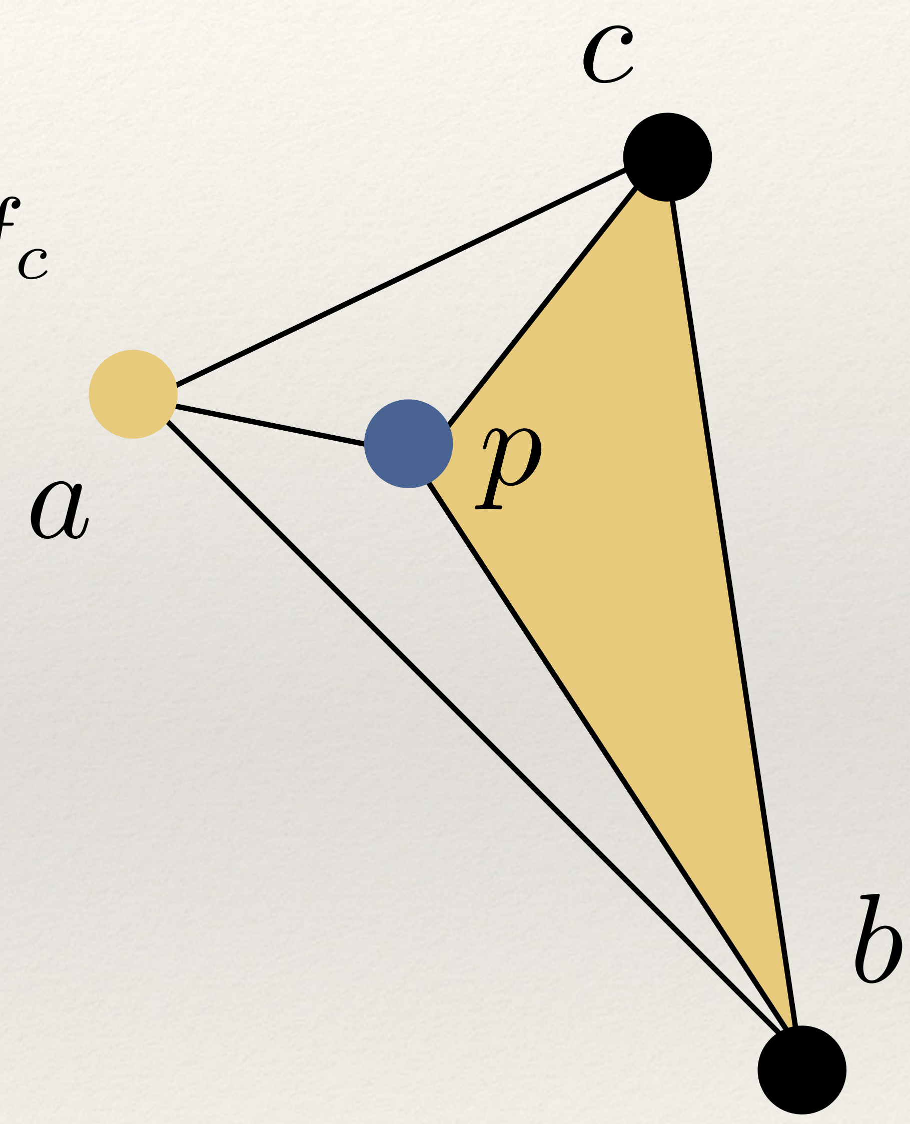
Barycentric Coordinates

$$f_p = f(\alpha, \beta, \gamma) = \alpha f_a + \beta f_b + \gamma f_c$$

$$\alpha = \frac{\text{area}(p, b, c)}{\text{area}(a, b, c)},$$

$$\beta = \frac{\text{area}(p, c, a)}{\text{area}(a, b, c)},$$

$$\gamma = \frac{\text{area}(p, a, b)}{\text{area}(a, b, c)}.$$



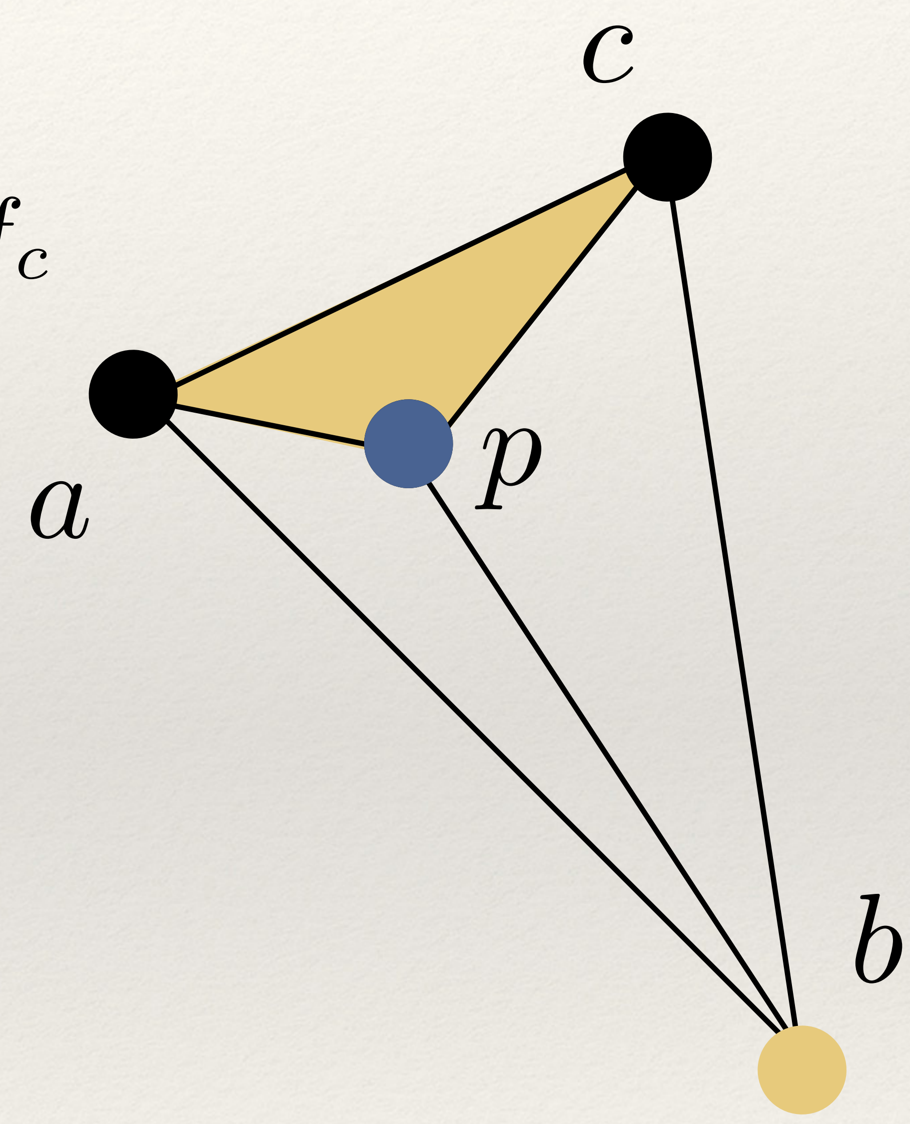
Barycentric Coordinates

$$f_p = f(\alpha, \beta, \gamma) = \alpha f_a + \beta f_b + \gamma f_c$$

$$\alpha = \frac{\text{area}(p, b, c)}{\text{area}(a, b, c)},$$

$$\beta = \frac{\text{area}(p, c, a)}{\text{area}(a, b, c)},$$

$$\gamma = \frac{\text{area}(p, a, b)}{\text{area}(a, b, c)}.$$



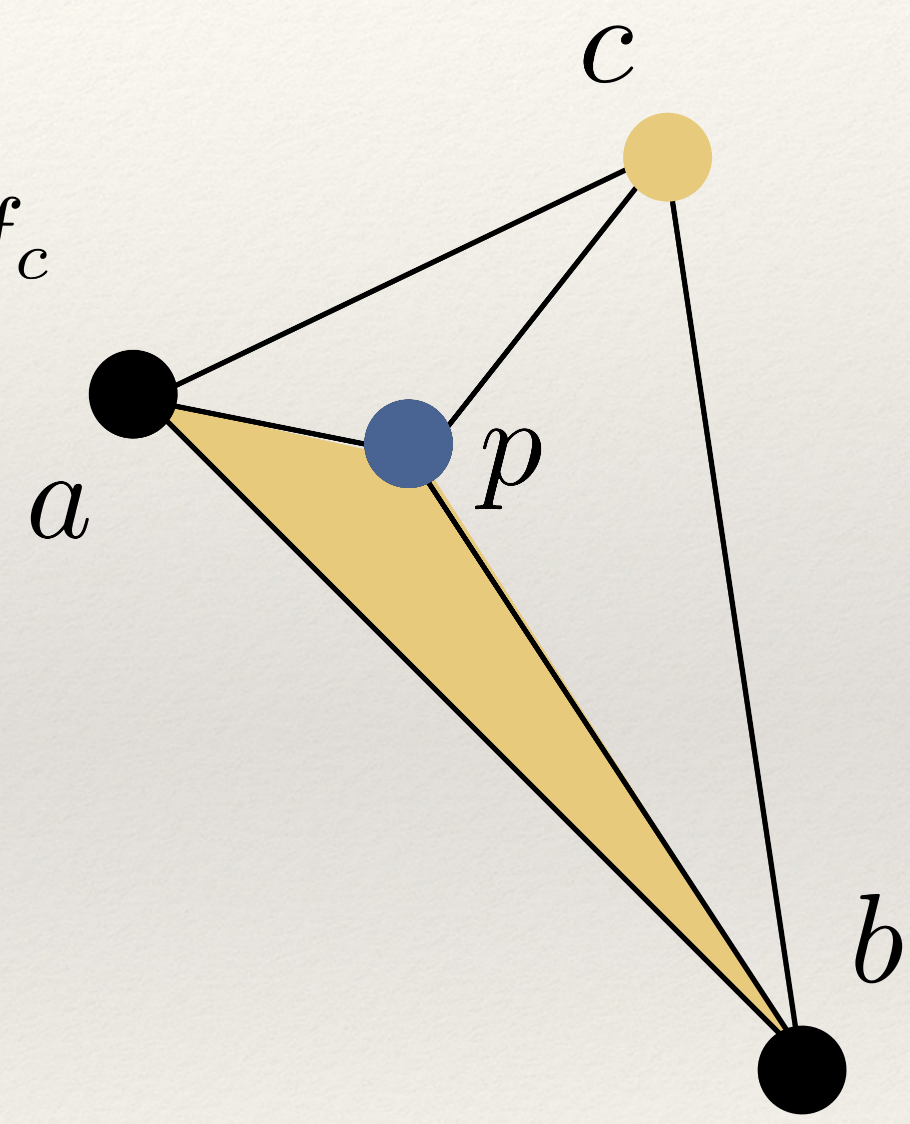
Barycentric Coordinates

$$f_p = f(\alpha, \beta, \gamma) = \alpha f_a + \beta f_b + \gamma f_c$$

$$\alpha = \frac{\text{area}(p, b, c)}{\text{area}(a, b, c)},$$

$$\beta = \frac{\text{area}(p, c, a)}{\text{area}(a, b, c)},$$

$$\gamma = \frac{\text{area}(p, a, b)}{\text{area}(a, b, c)}.$$



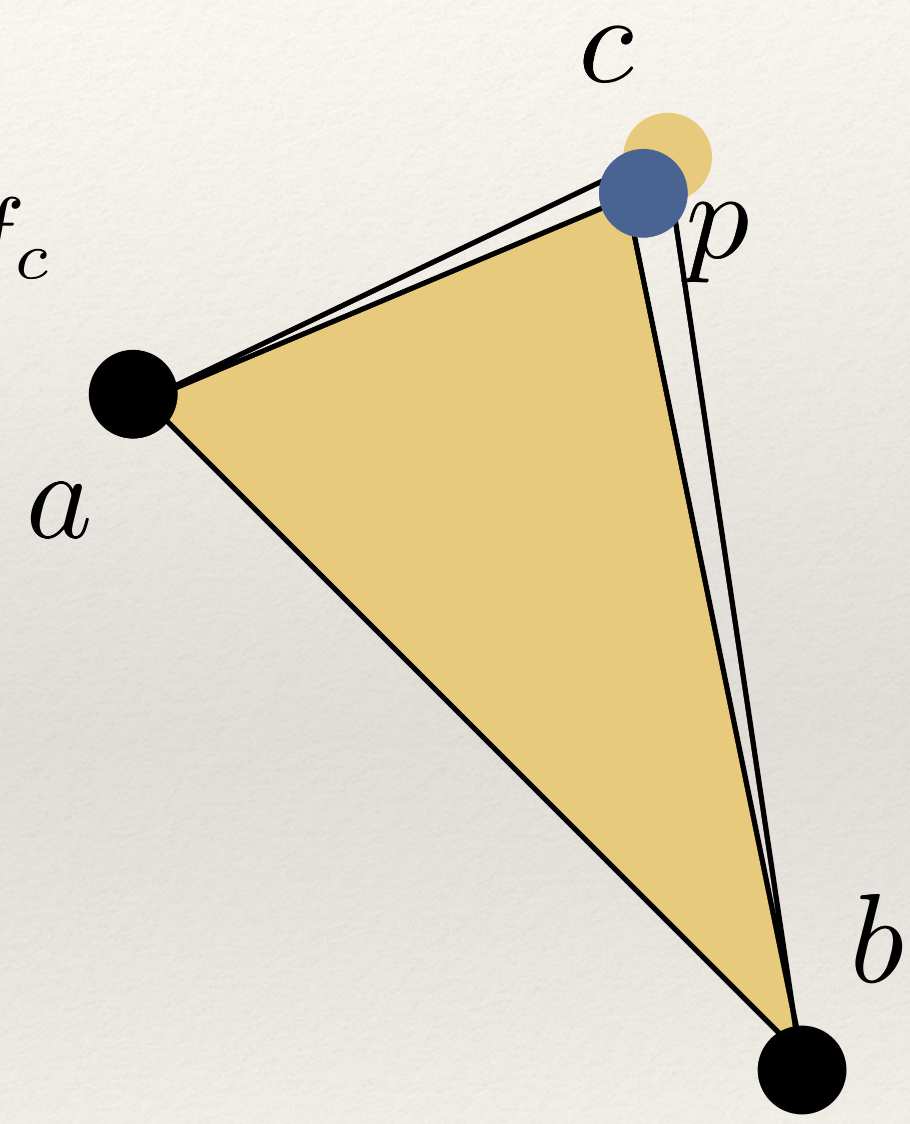
Barycentric Coordinates

$$f_p = f(\alpha, \beta, \gamma) = \alpha f_a + \beta f_b + \gamma f_c$$

$$\alpha = \frac{\text{area}(p, b, c)}{\text{area}(a, b, c)},$$

$$\beta = \frac{\text{area}(p, c, a)}{\text{area}(a, b, c)},$$

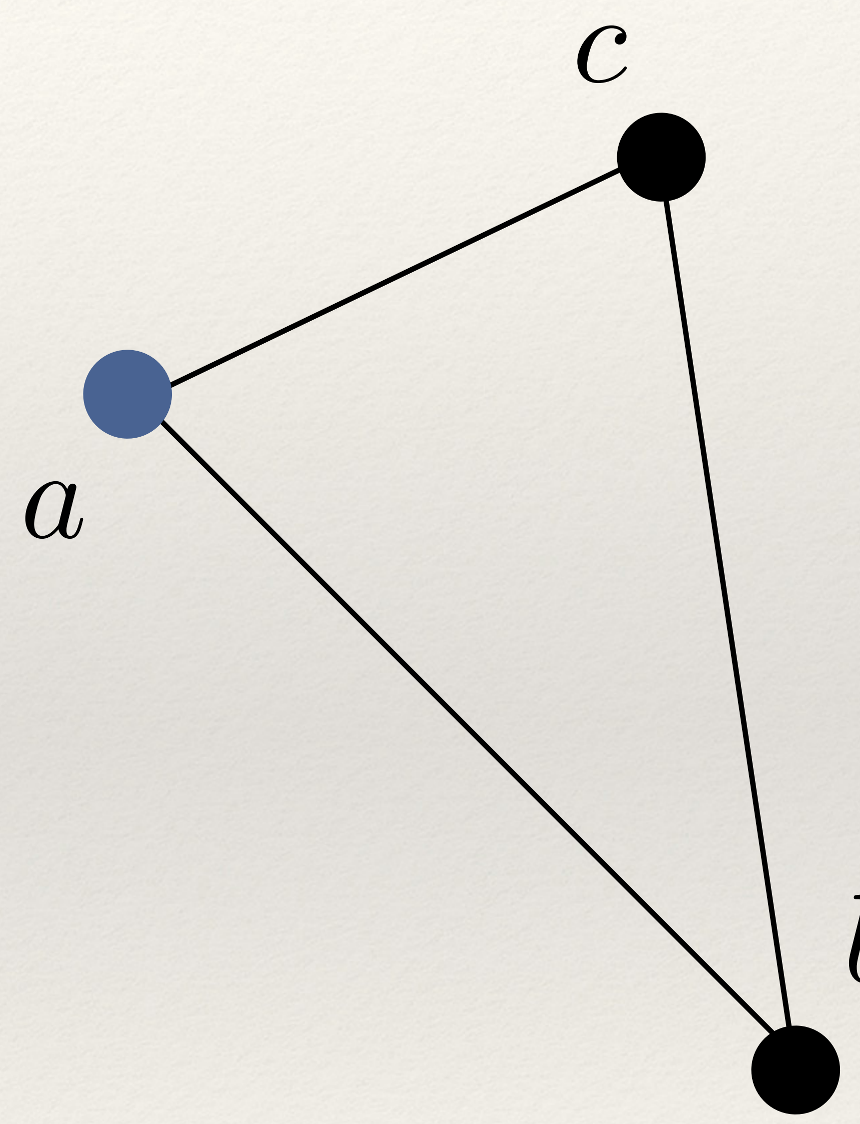
$$\gamma = \frac{\text{area}(p, a, b)}{\text{area}(a, b, c)}.$$



Barycentric Coordinates

- Coordinates at a vertex

$$\alpha = 1$$
$$\beta = 0$$
$$\gamma = 0$$



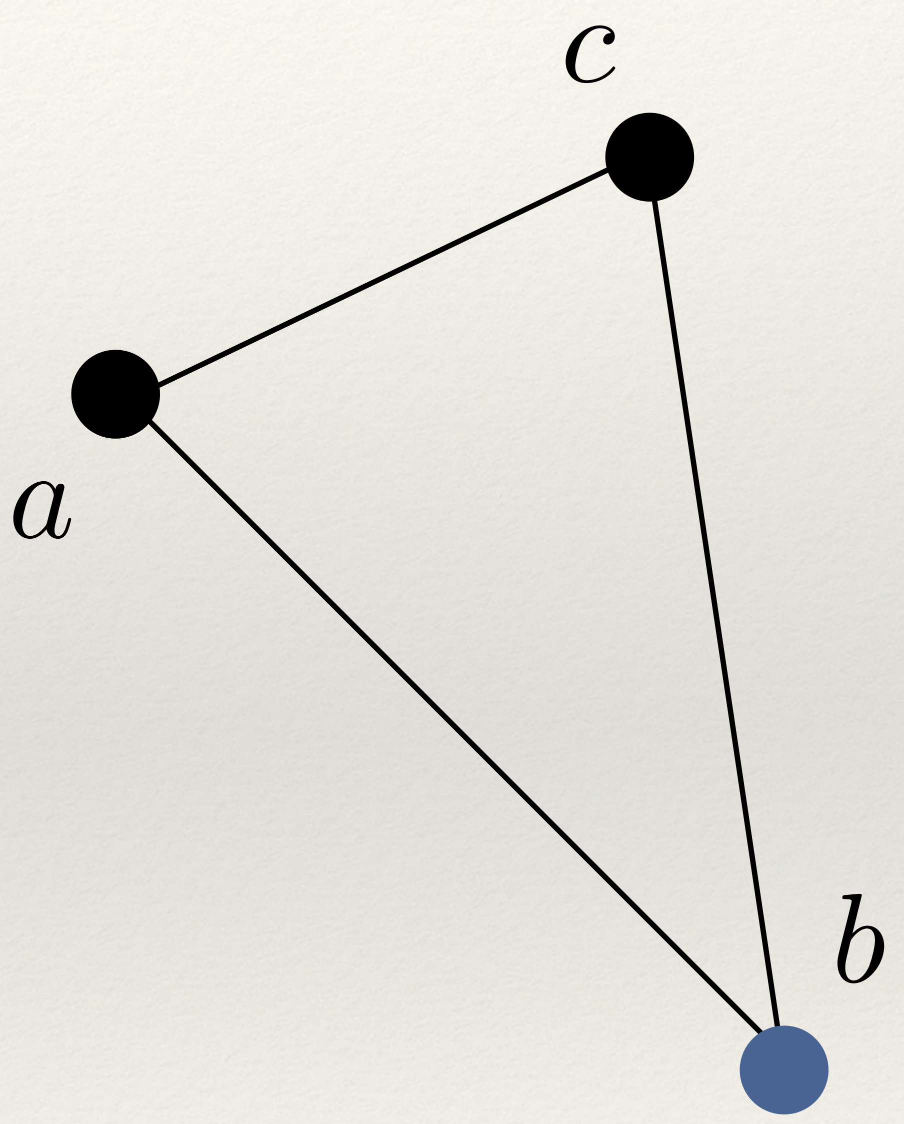
Barycentric Coordinates

- Coordinates at a vertex

$$\alpha = 0$$

$$\beta = 1$$

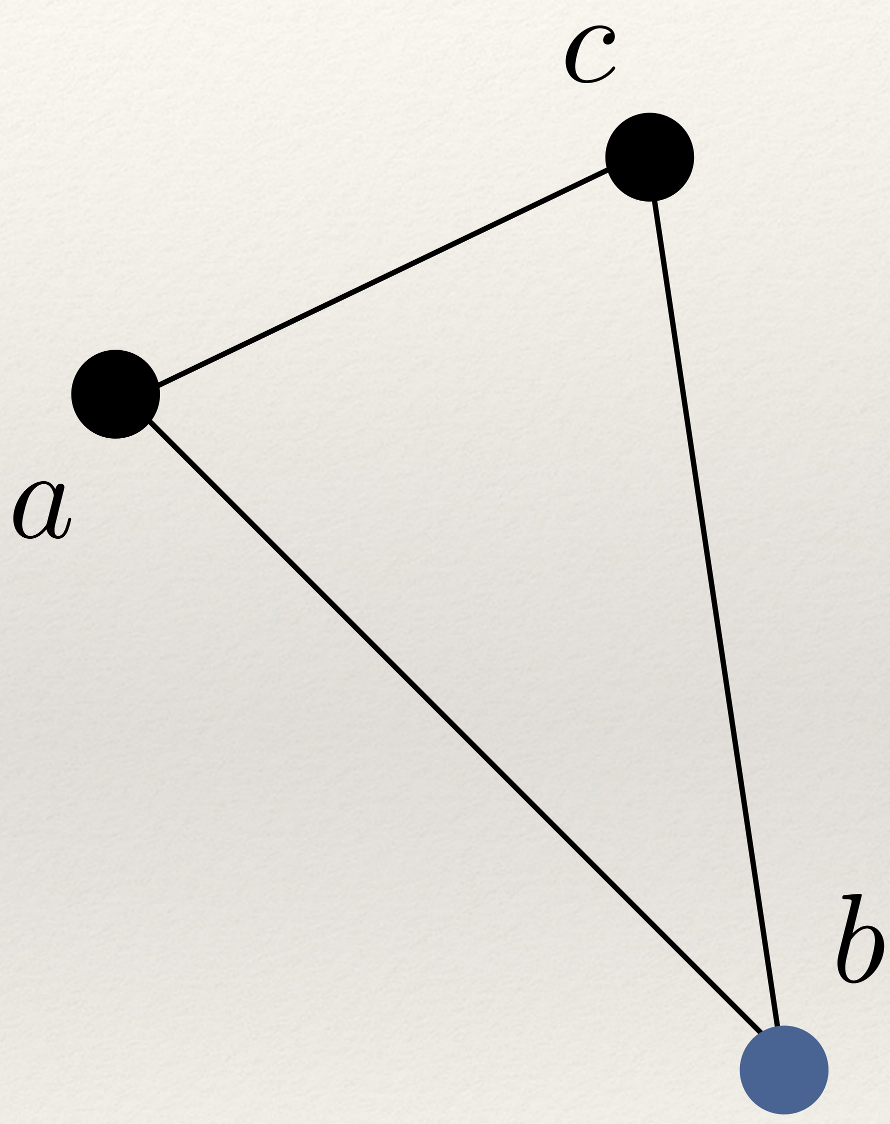
$$\gamma = 0$$



Barycentric Coordinates

- Coordinates at a vertex

$$\alpha = 0$$
$$\beta = 0$$
$$\gamma = 1$$



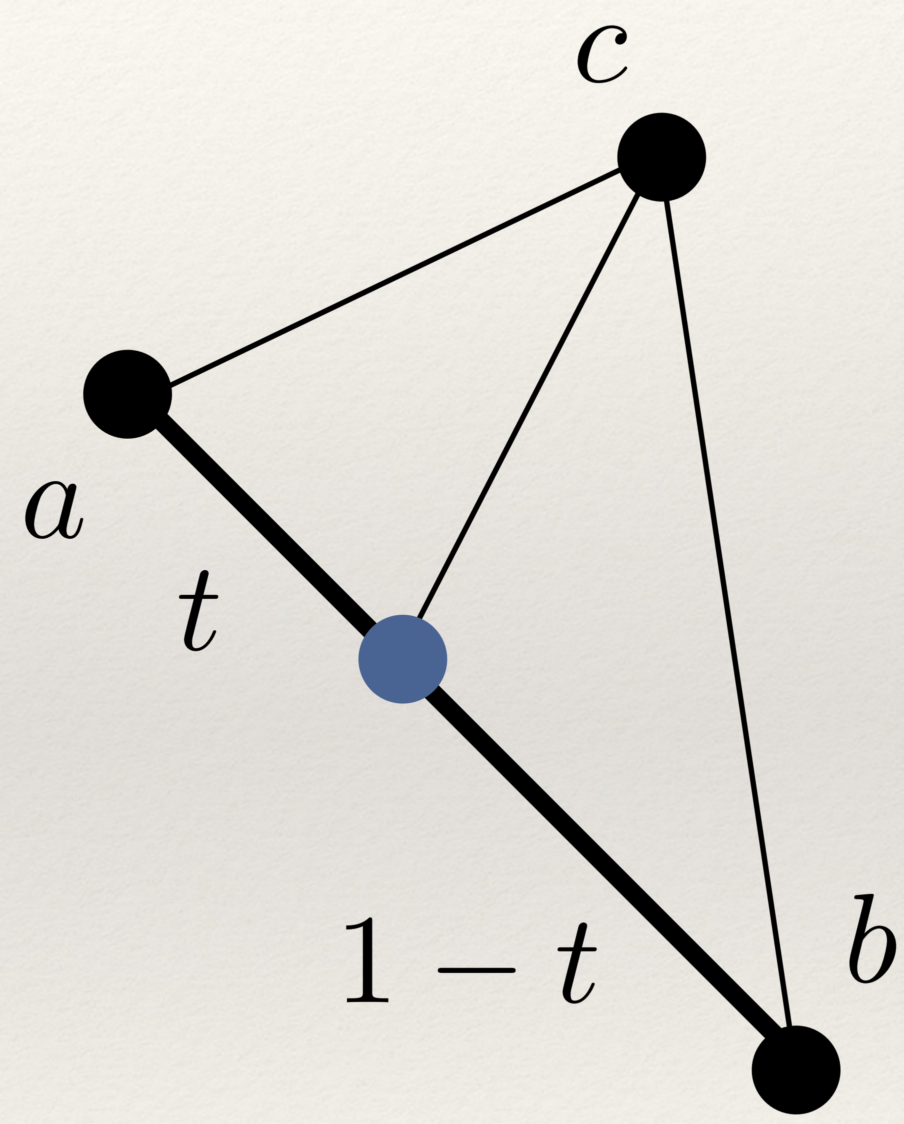
Barycentric Coordinates

- Coordinates on edge

$$\alpha = 1 - t$$

$$\beta = t$$

$$\gamma = 0$$



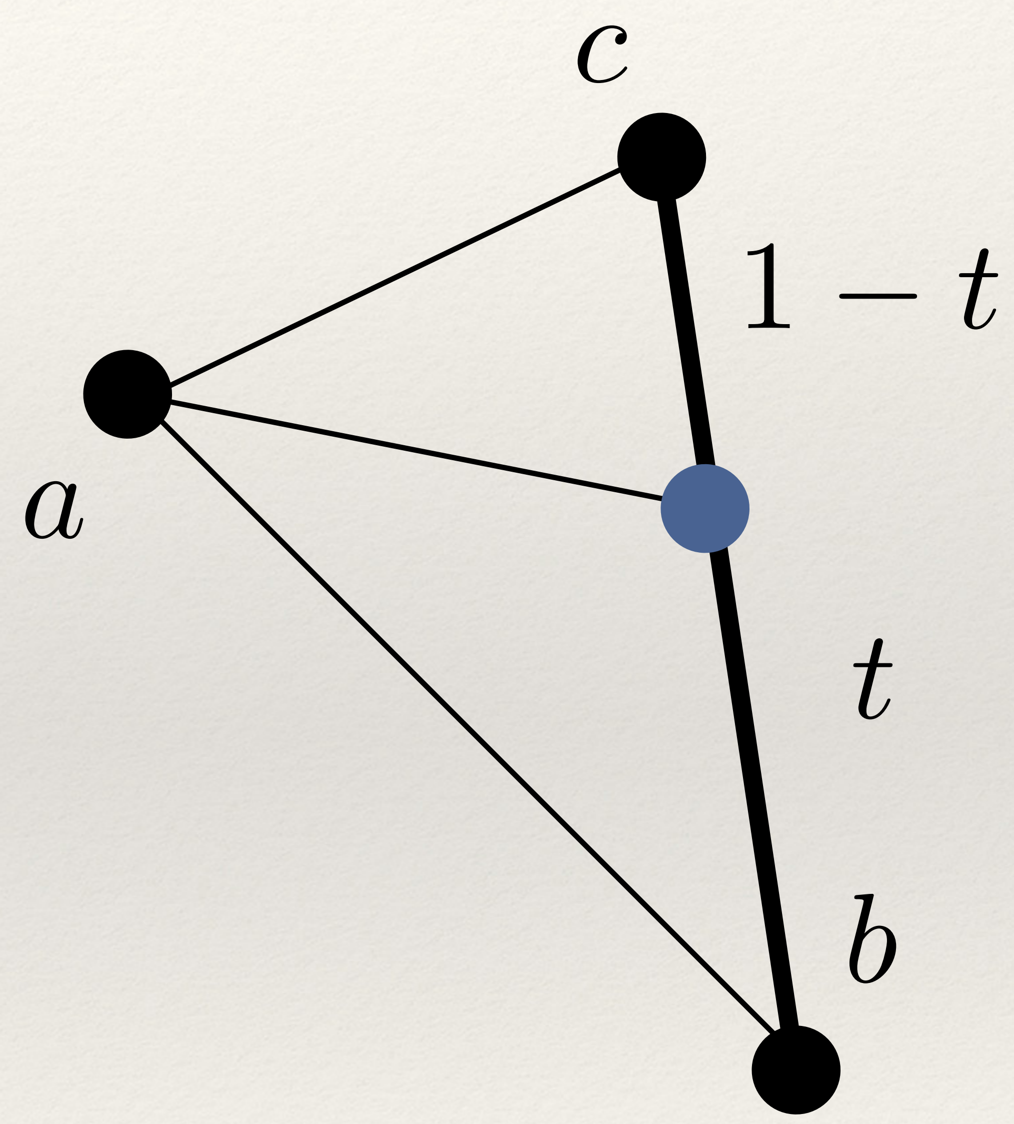
Barycentric Coordinates

- Coordinates on edge

$$\alpha = 0$$

$$\beta = 1 - t$$

$$\gamma = t$$



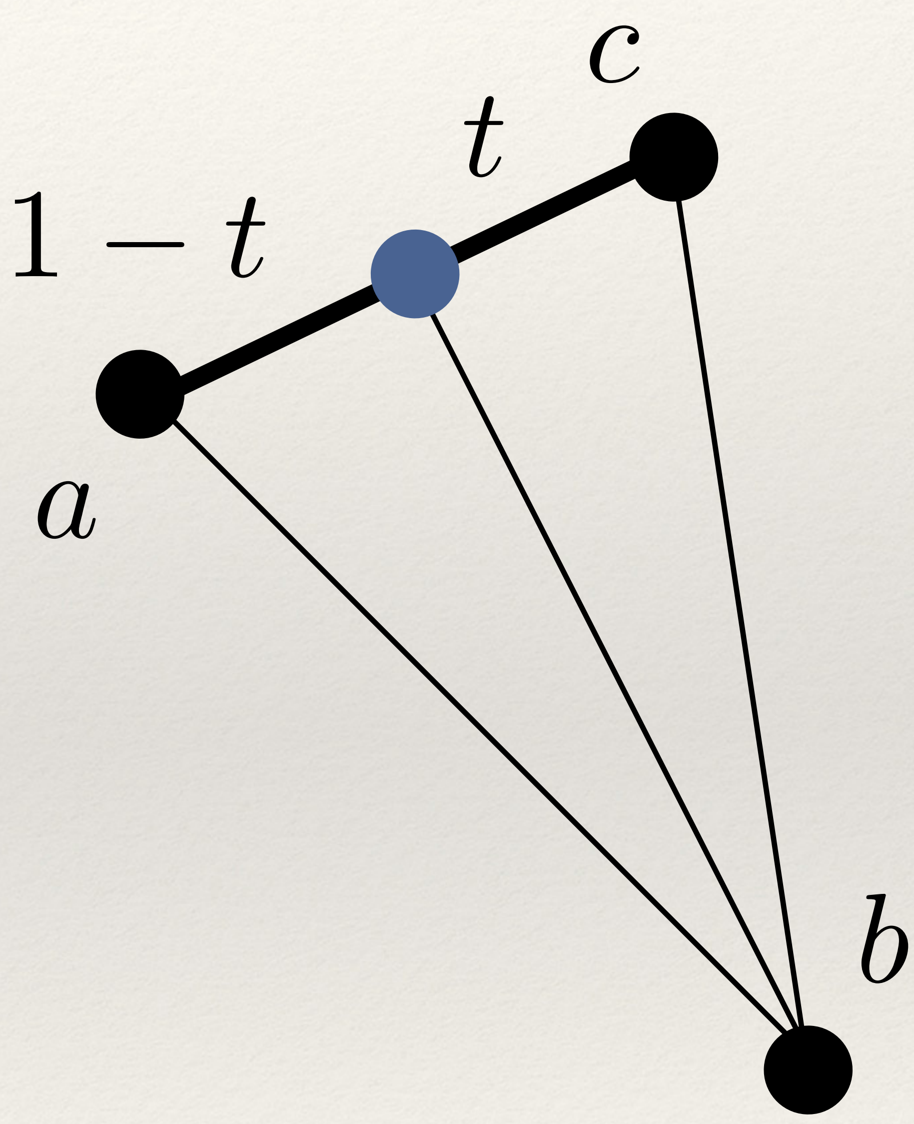
Barycentric Coordinates

- Coordinates on edge

$$\alpha = t$$

$$\beta = 0$$

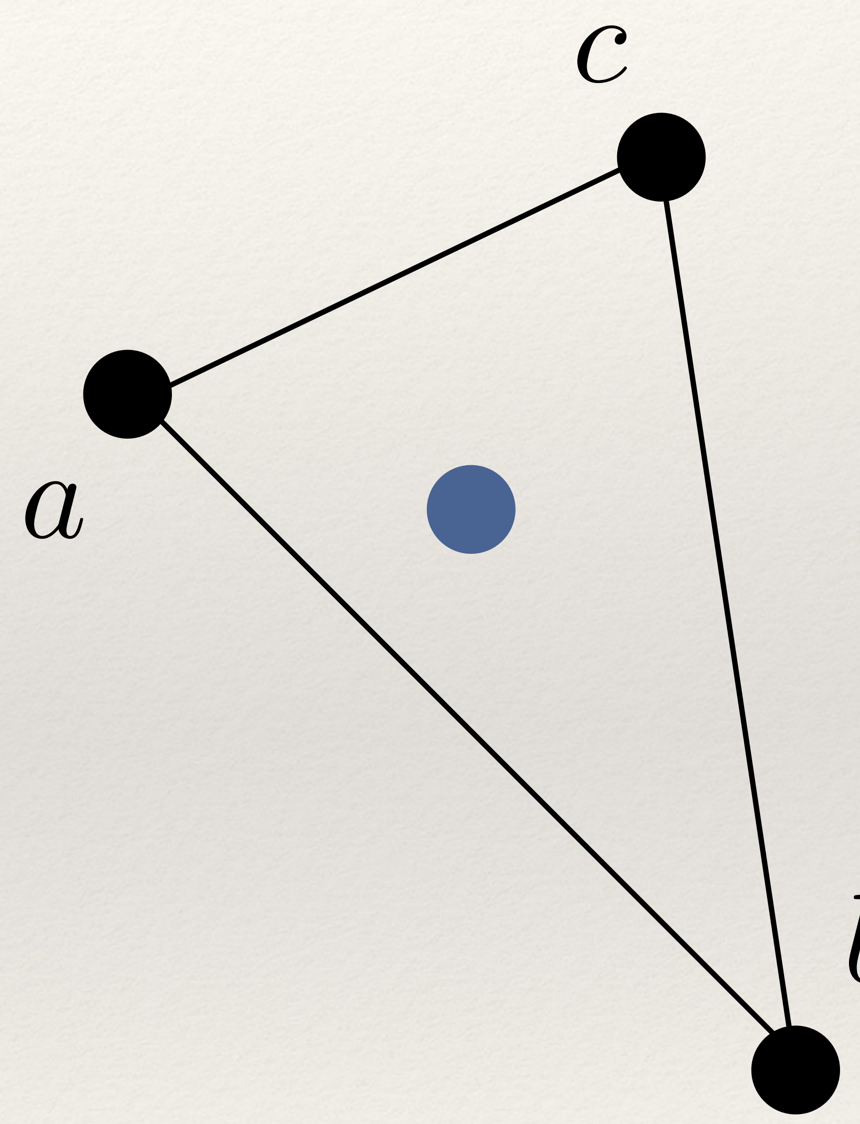
$$\gamma = 1 - t$$



Barycentric Coordinates

- Inside/outside test

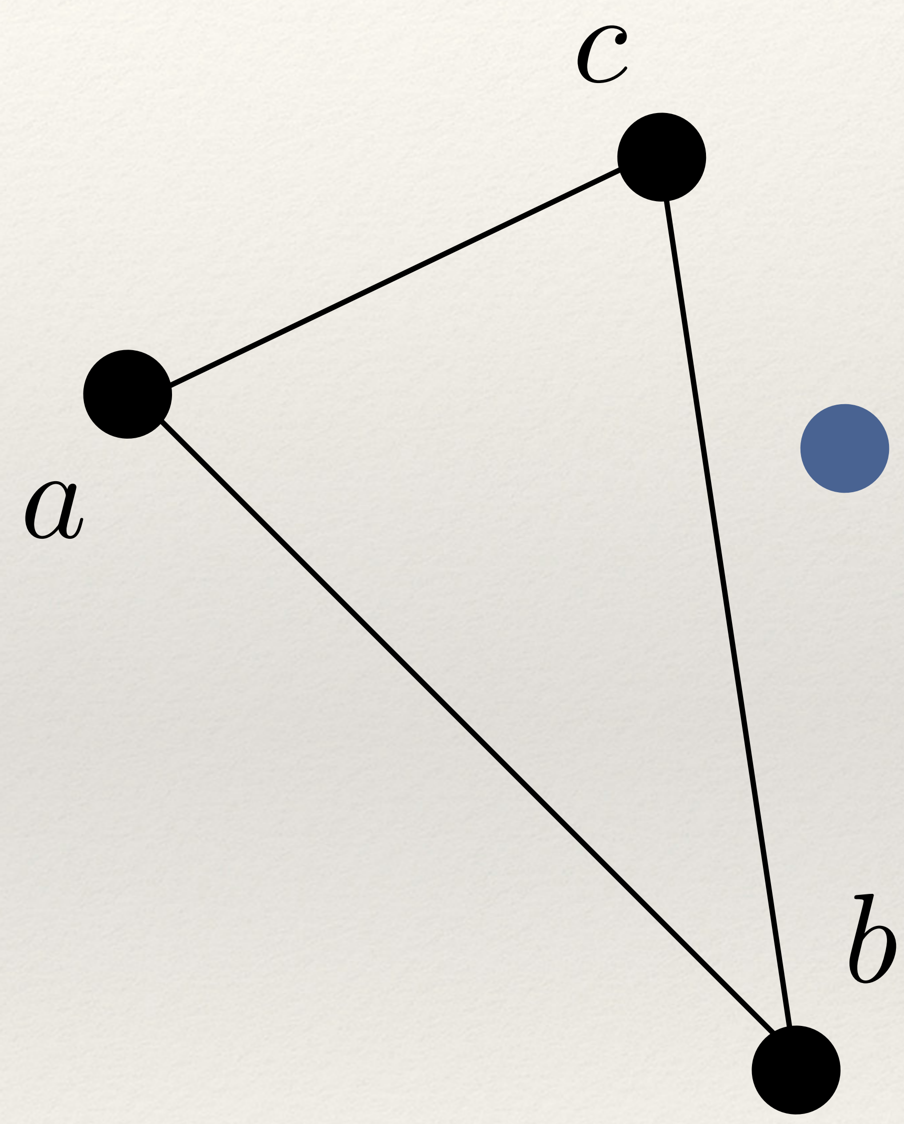
$$\alpha > 0, \beta > 0, \gamma > 0$$



Barycentric Coordinates

- Inside/outside test

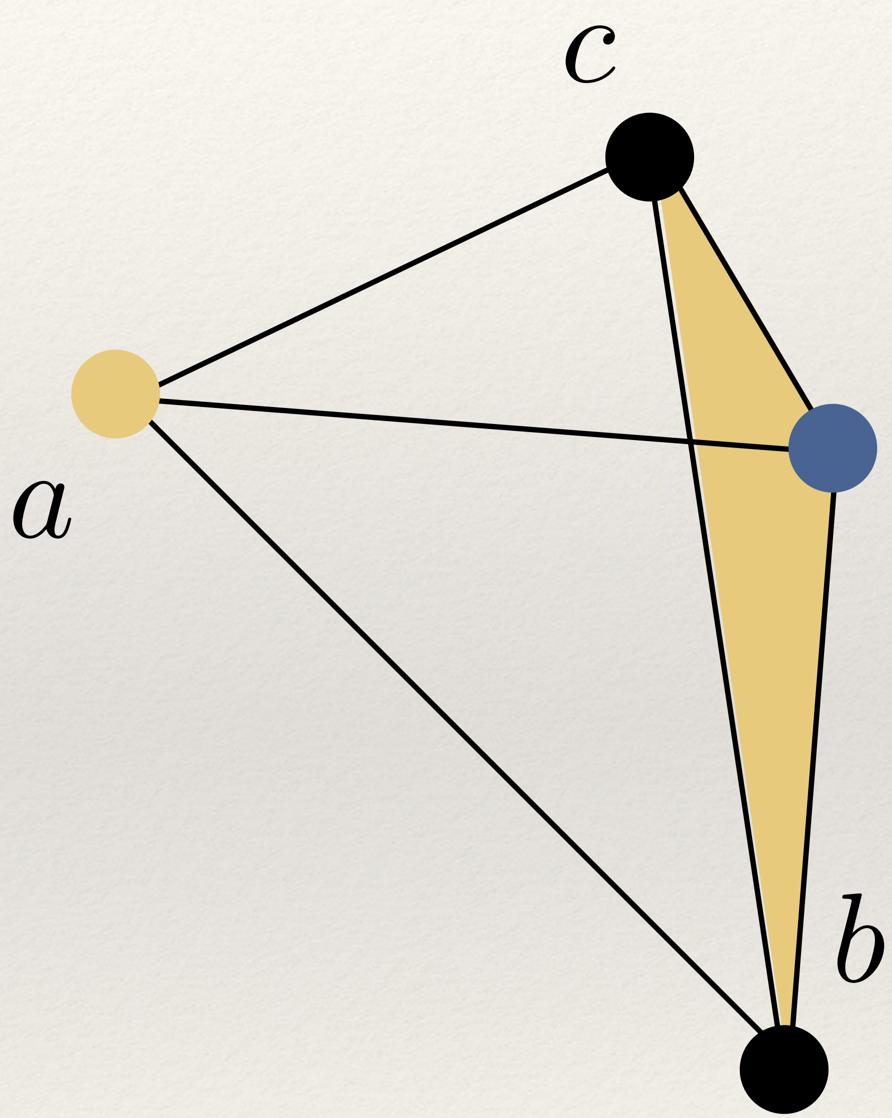
$$\alpha < 0, \beta > 0, \gamma > 0$$



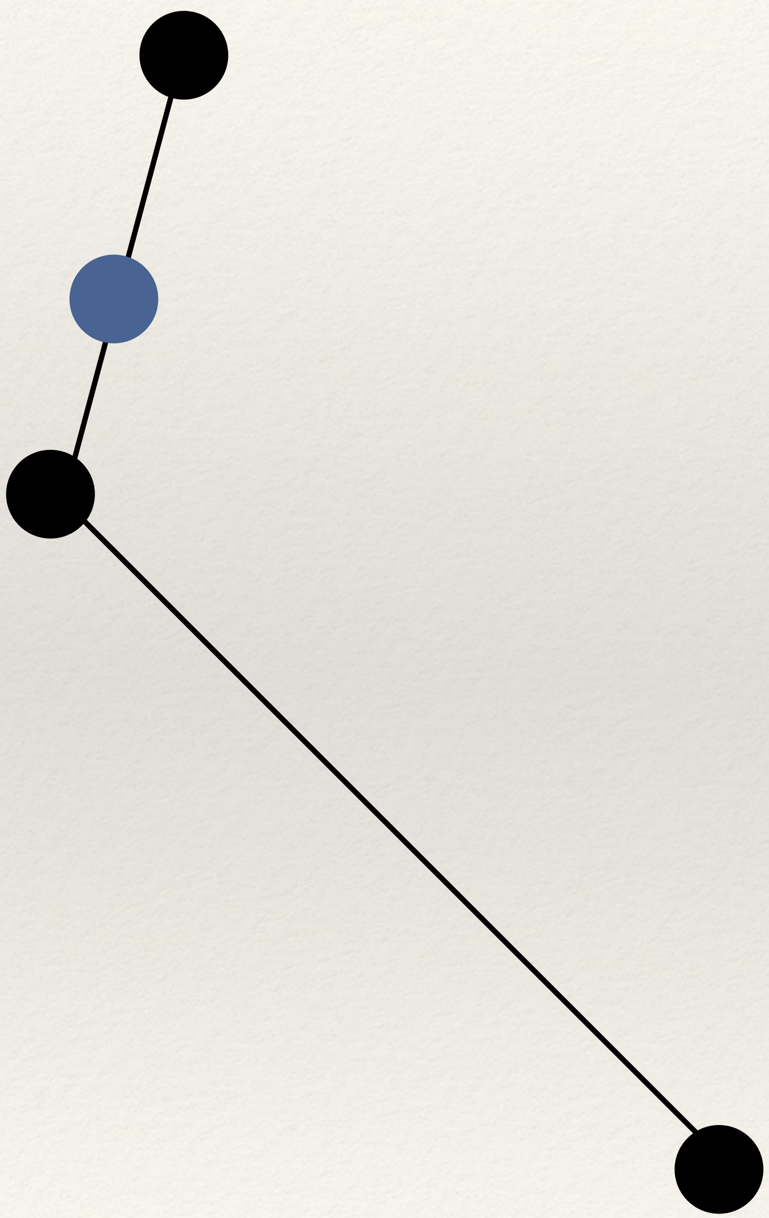
Barycentric Coordinates

- Inside/outside test

$$\alpha < 0, \beta > 0, \gamma > 0$$

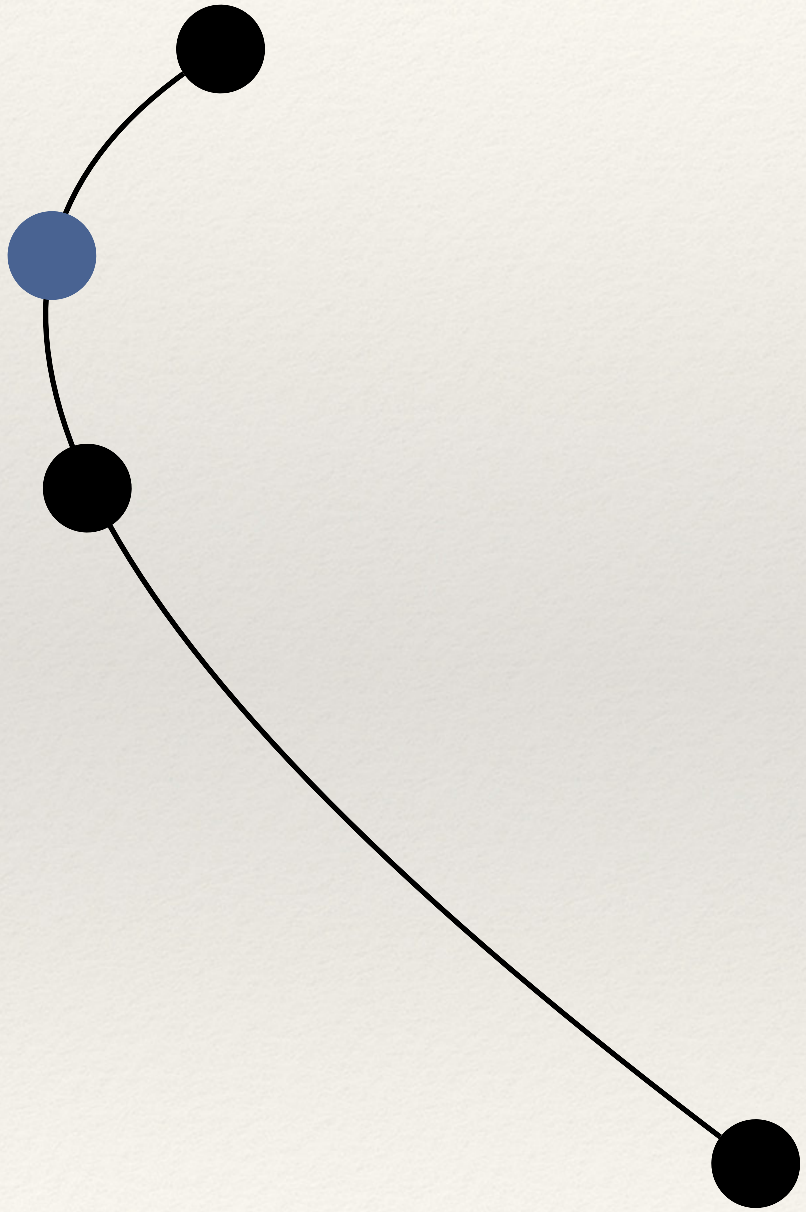


Polynomial Interpolation



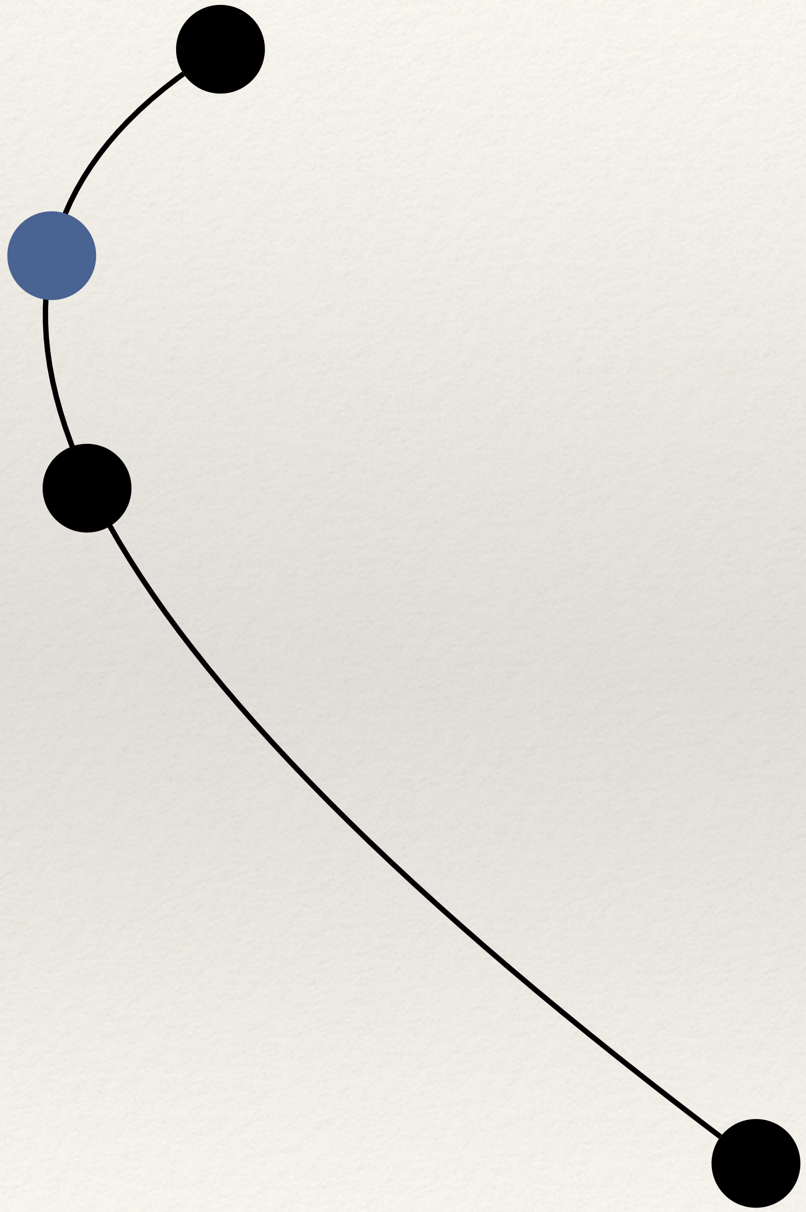
$$f(t) = at + b$$

Polynomial Interpolation



$$f(t) = at^2 + bt + c$$

Polynomial Interpolation



- Lagrange interpolation
- Newton interpolation
- Same polynomial
- Different cost of construction and evaluation

Other Methods

- Bezier curves, splines
- Harmonic coordinates, mean value coordinates, Green coordinates

Finite Differences

Finite Difference Methods

- Used to discretize spatial derivatives

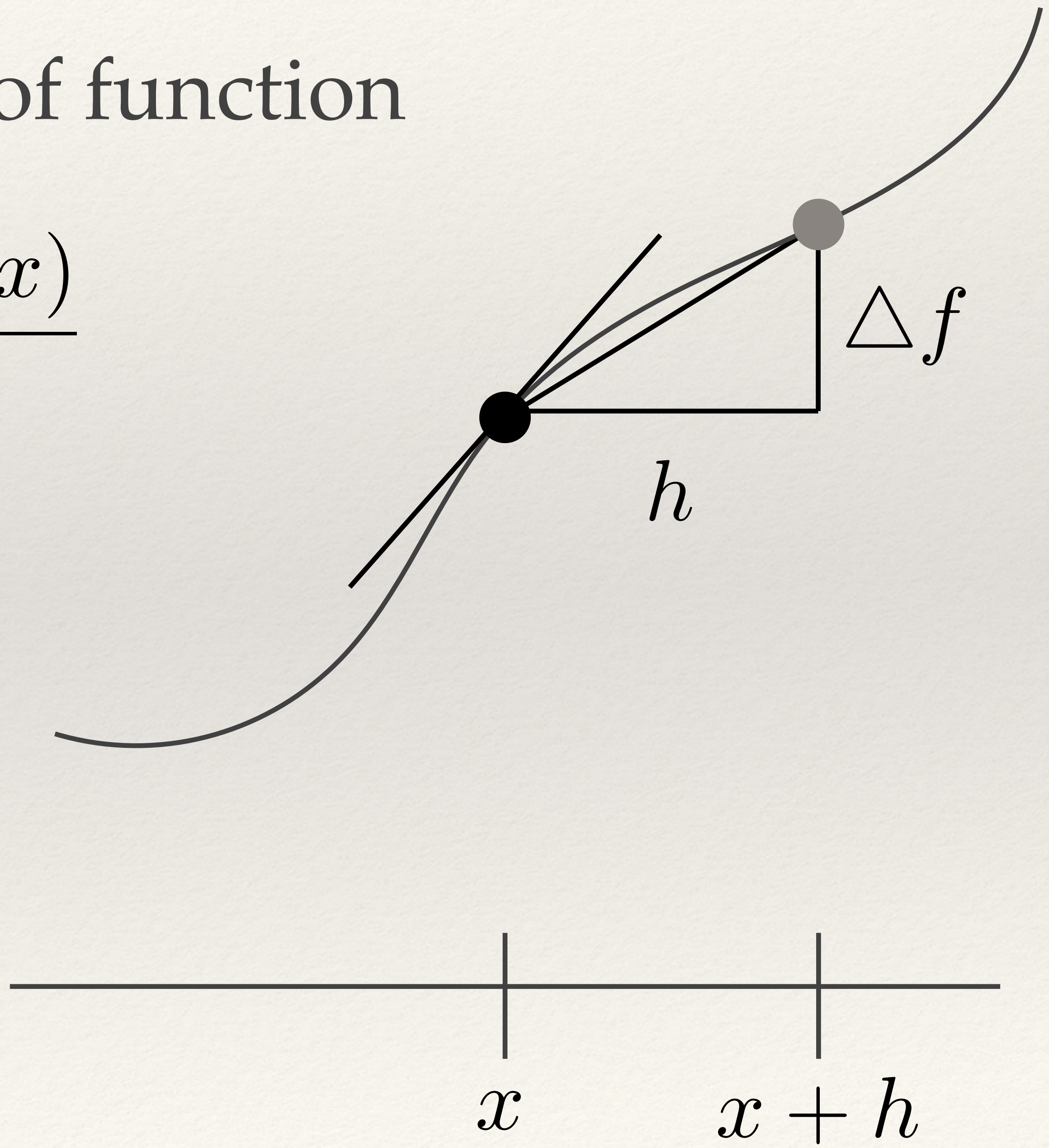
$$\rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \mu \Delta \mathbf{u} + \mathbf{f}$$

$$\nabla \cdot \mathbf{u} = 0$$

Finite Difference Methods

- Recall definition of derivative of function

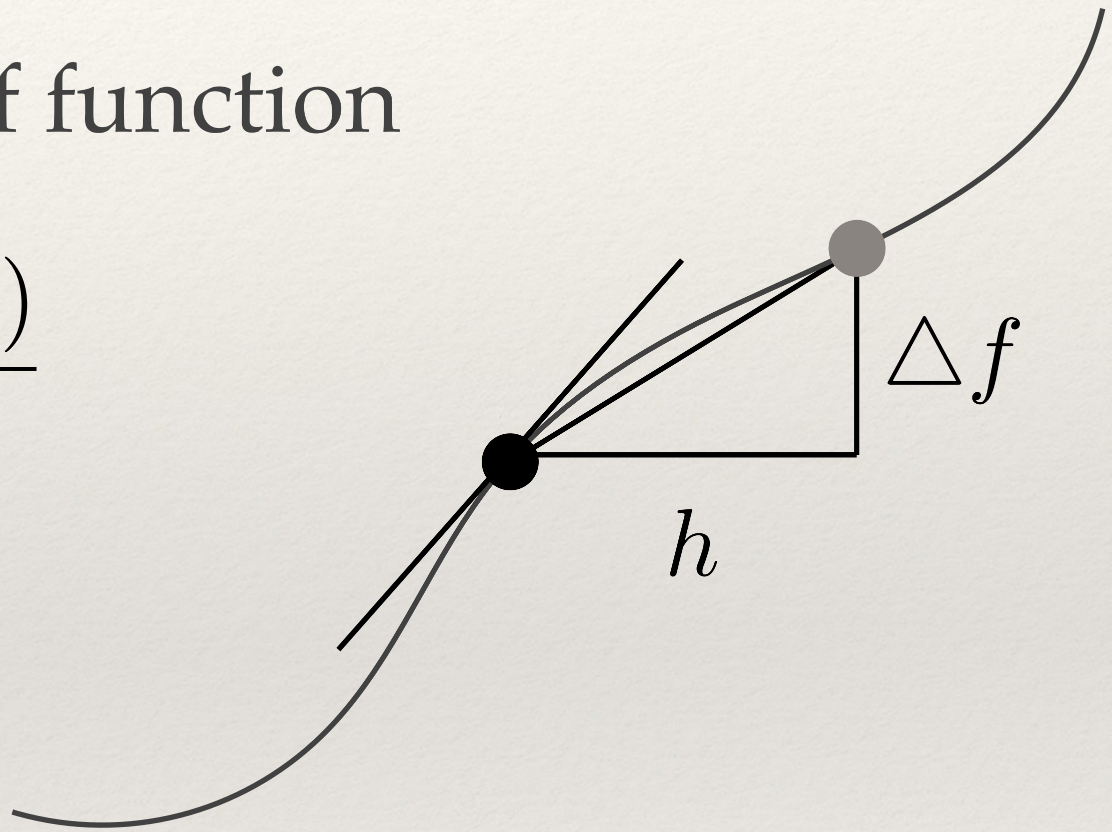
$$\frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$



Finite Difference Methods

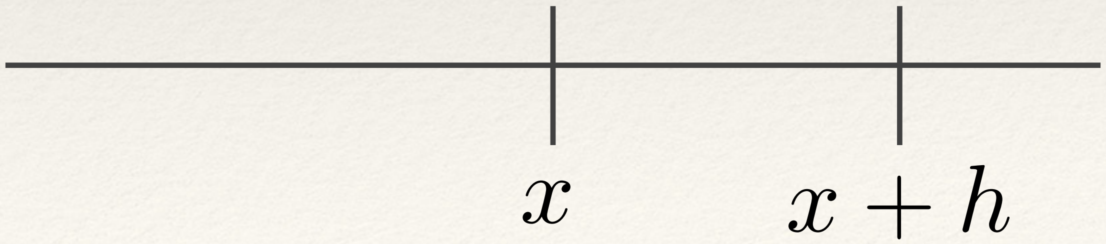
- Recall definition of derivative of function

$$\frac{d}{dx}f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



- Finite difference approximation

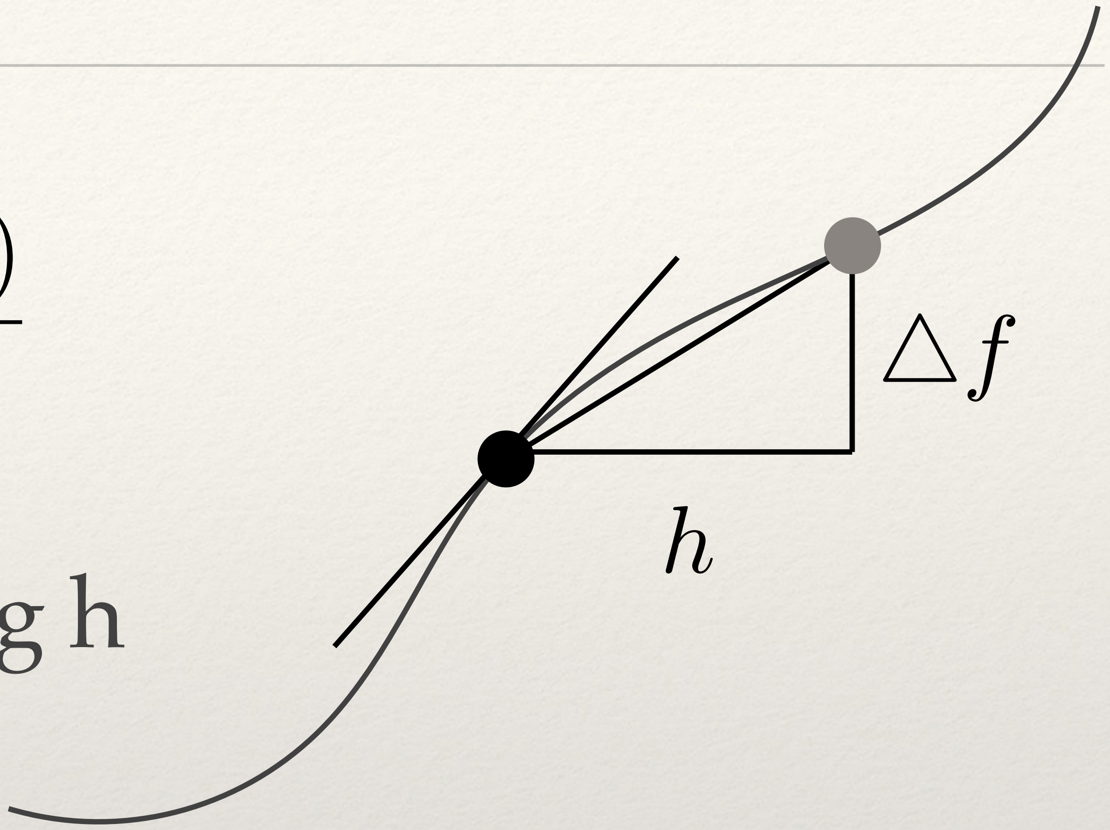
$$\frac{d}{dx}f(x) \approx \frac{f(x+h) - f(x)}{h}$$



forward difference

Finite Difference Methods

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$



- Error decreases with decreasing h
- Taylor series

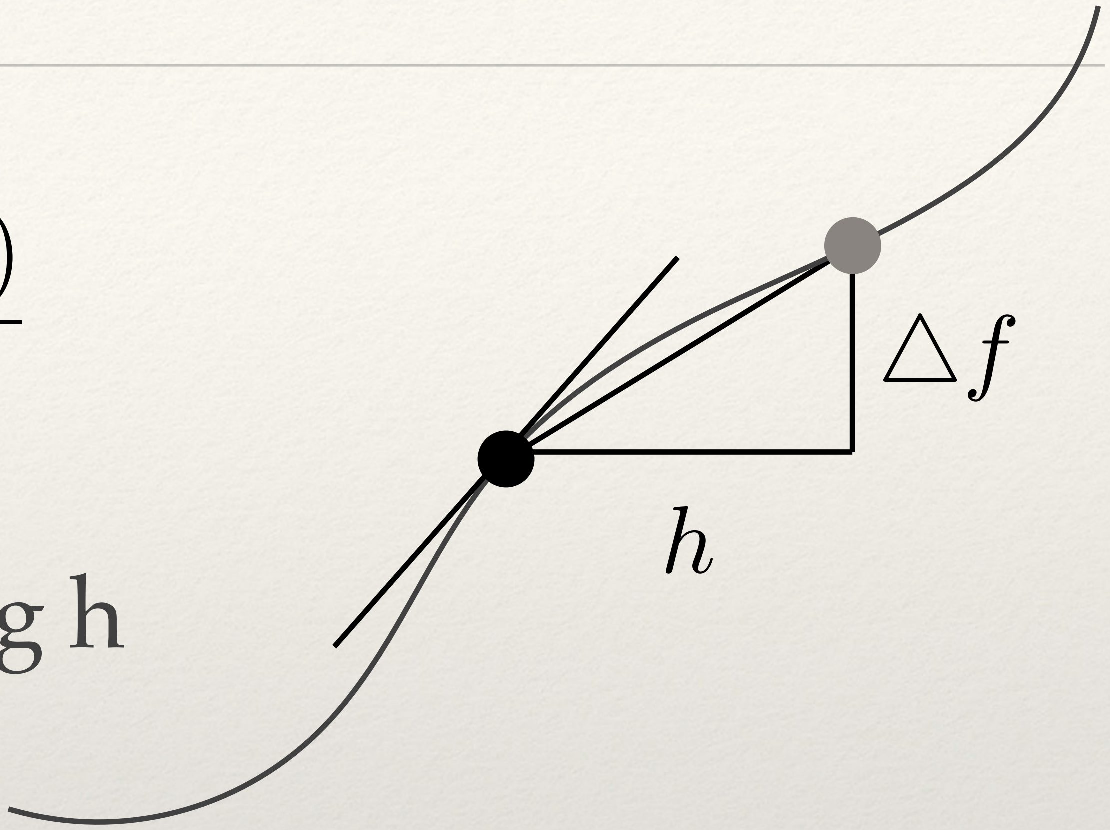
$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \ldots + \frac{h^n}{n!}f^{(n)}(x) + \ldots$$

- Rearrange

$$\frac{f(x+h) - f(x)}{h} = f'(x) + \frac{h}{2}f''(x) + \ldots$$

Finite Difference Methods

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$



- Error decreases with decreasing h
- Taylor series

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \ldots + \frac{h^n}{n!} f^{(n)}(x) + \ldots$$

- Rearrange

$$\frac{f(x+h) - f(x)}{h} = f'(x) + O(h)$$

first order accurate

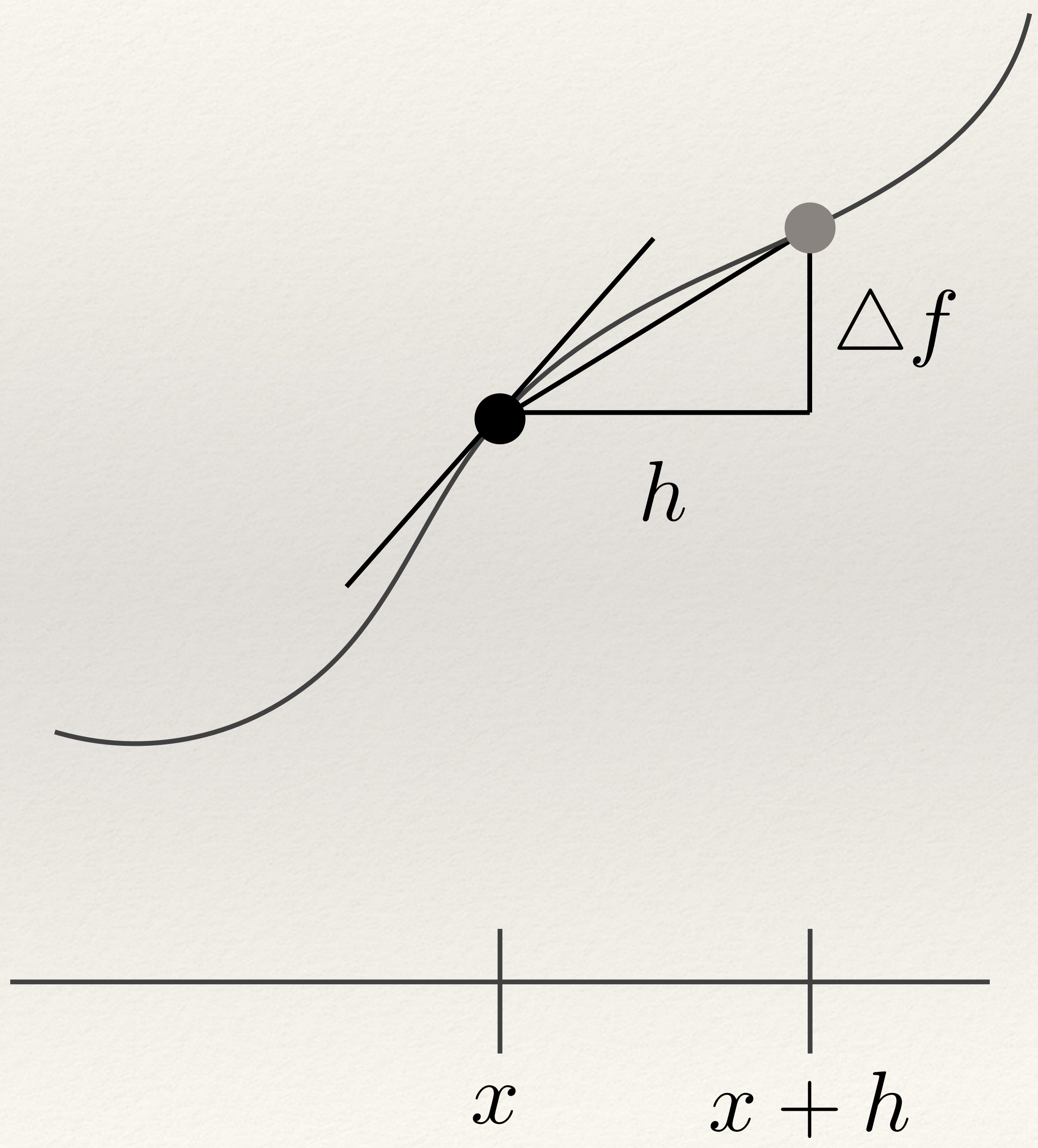
Forward and Backward Difference

- Forward difference

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

- Backward difference

$$f'(x) \approx \frac{f(x) - f(x-h)}{h}$$



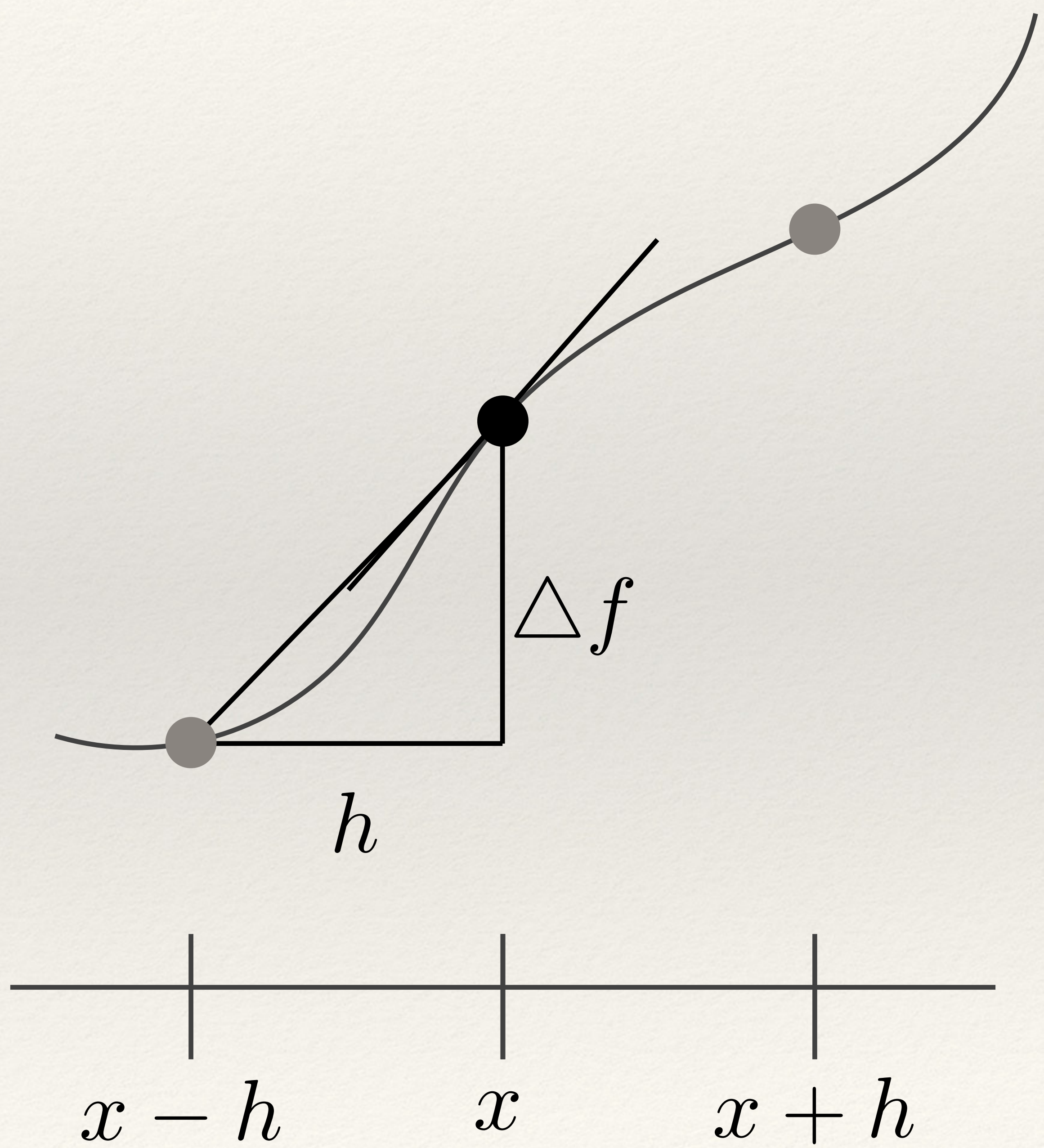
Forward and Backward Difference

- Forward difference

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

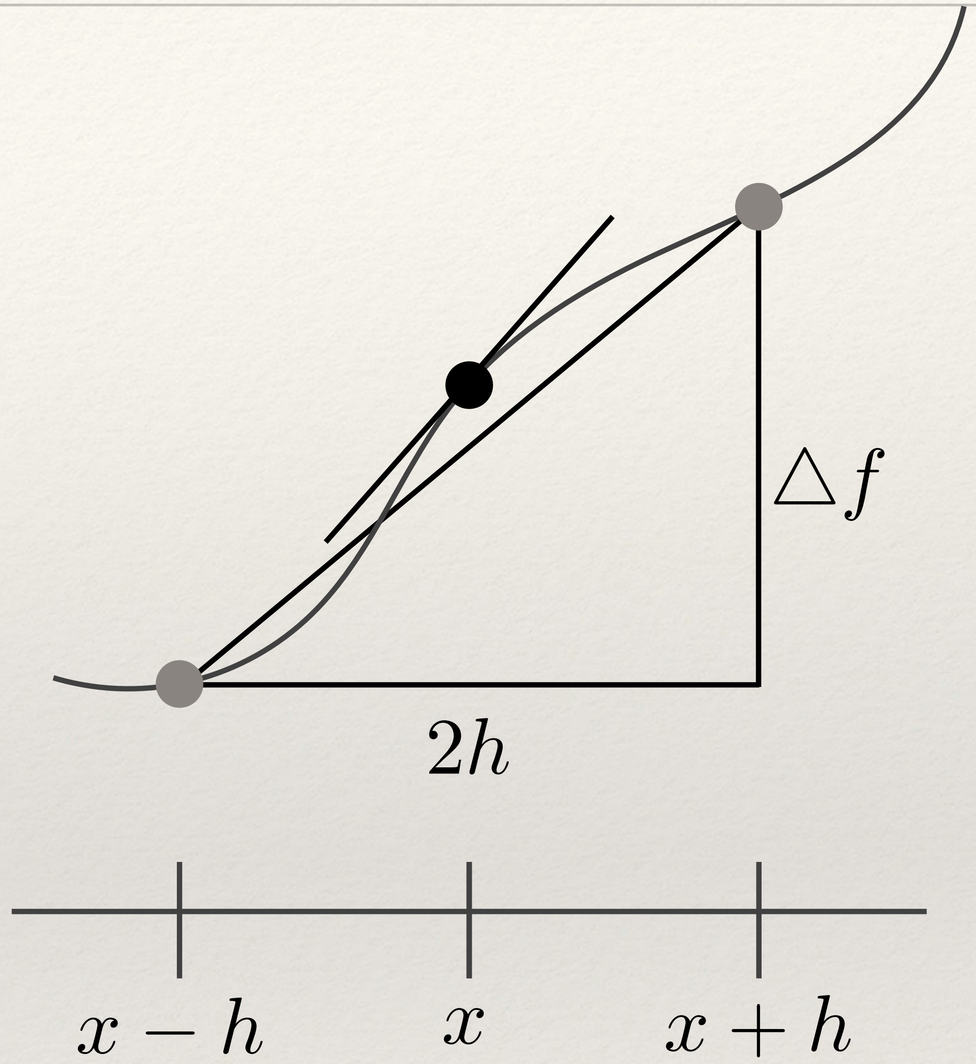
- Backward difference

$$f'(x) \approx \frac{f(x) - f(x-h)}{h}$$



Central Difference Approximation

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$



- Order of accuracy

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + O(h^2)$$

second order accurate

Discretization Error

- Forward difference

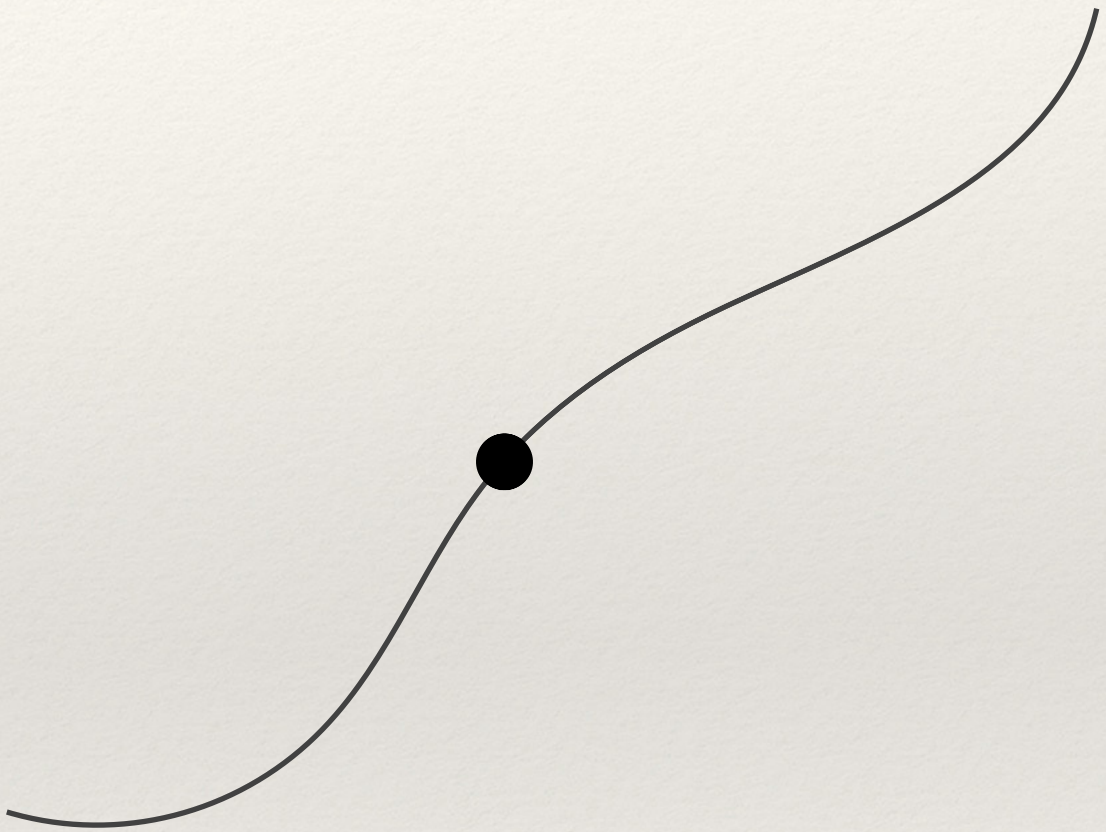
$$\frac{f(x+h)-f(x)}{h} = f'(x) + \frac{h}{2}f''(x) + \ldots$$

- Central difference

$$\frac{f(x+h)-f(x-h)}{2h} = f'(x) + \frac{h^2}{6}f'''(x) + \ldots$$

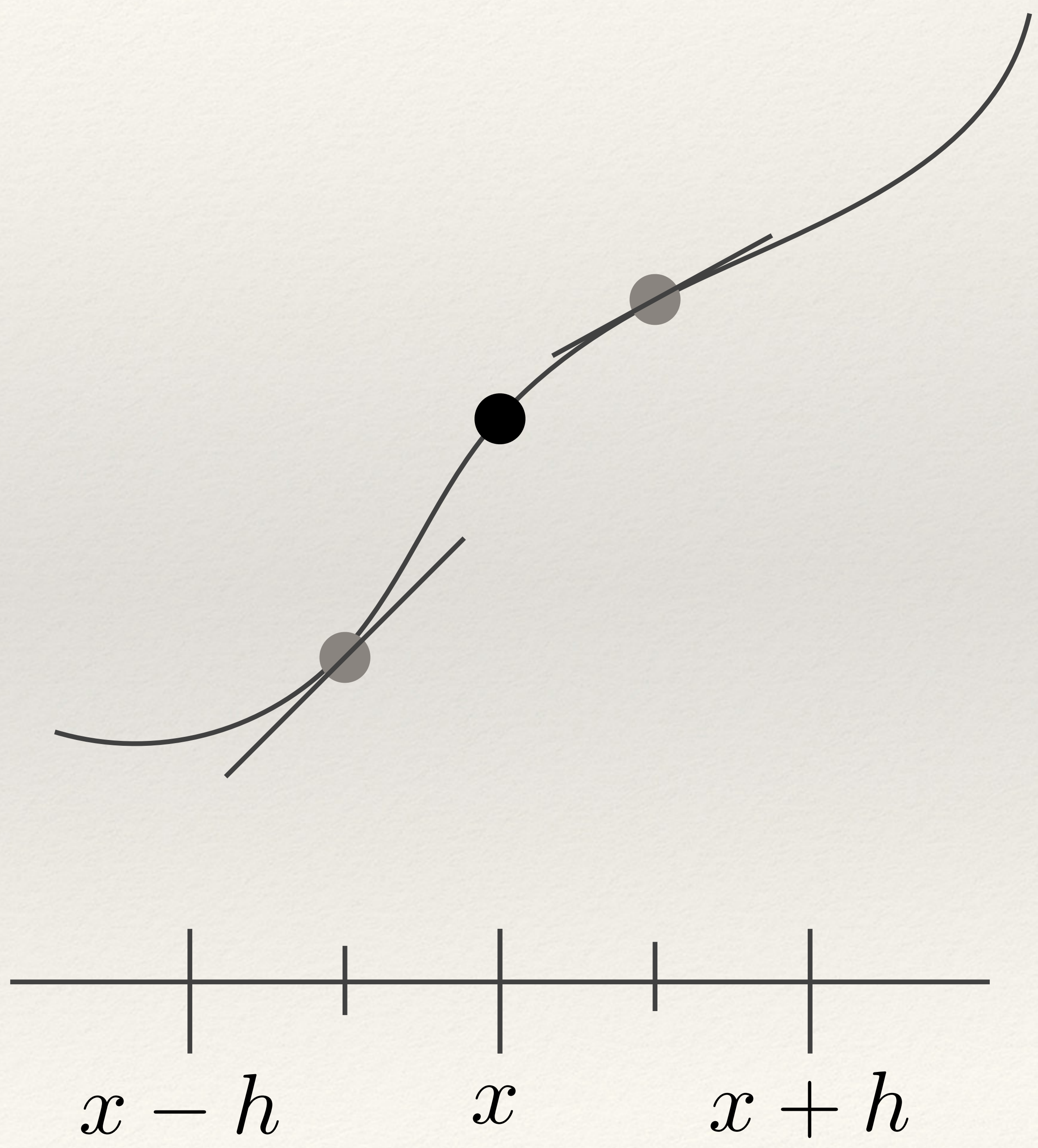
Higher Derivatives

$f''(x)$



Higher Derivatives

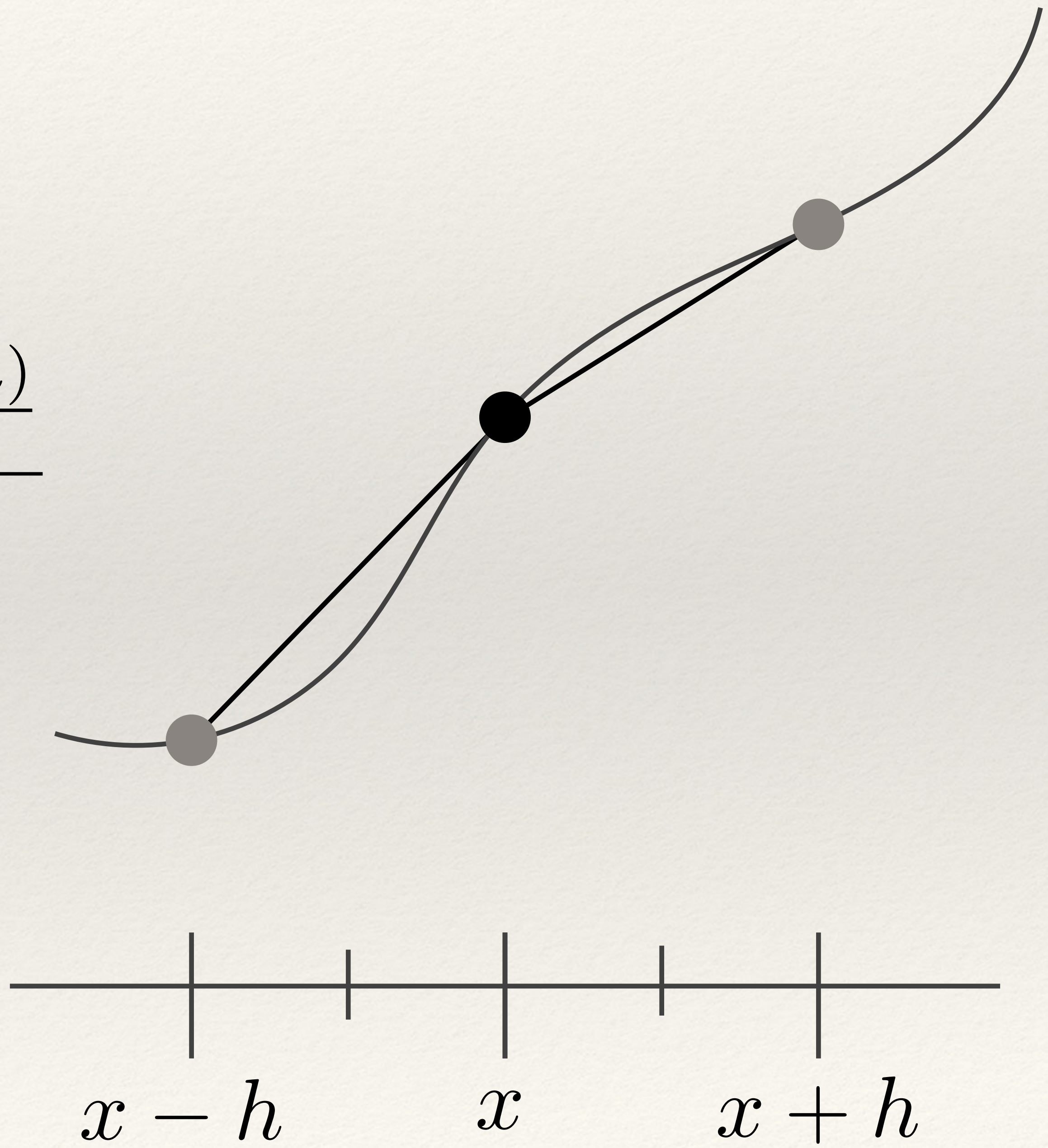
$$f''(x) \approx \frac{f'(x+\frac{h}{2}) - f'(x-\frac{h}{2})}{h}$$



Higher Derivatives

$$f''(x) \approx \frac{f'(x+\frac{h}{2}) - f'(x-\frac{h}{2})}{h}$$

$$\approx \frac{\frac{f(x+h)-f(x)}{h} - \frac{f(x)-f(x-h)}{h}}{h}$$

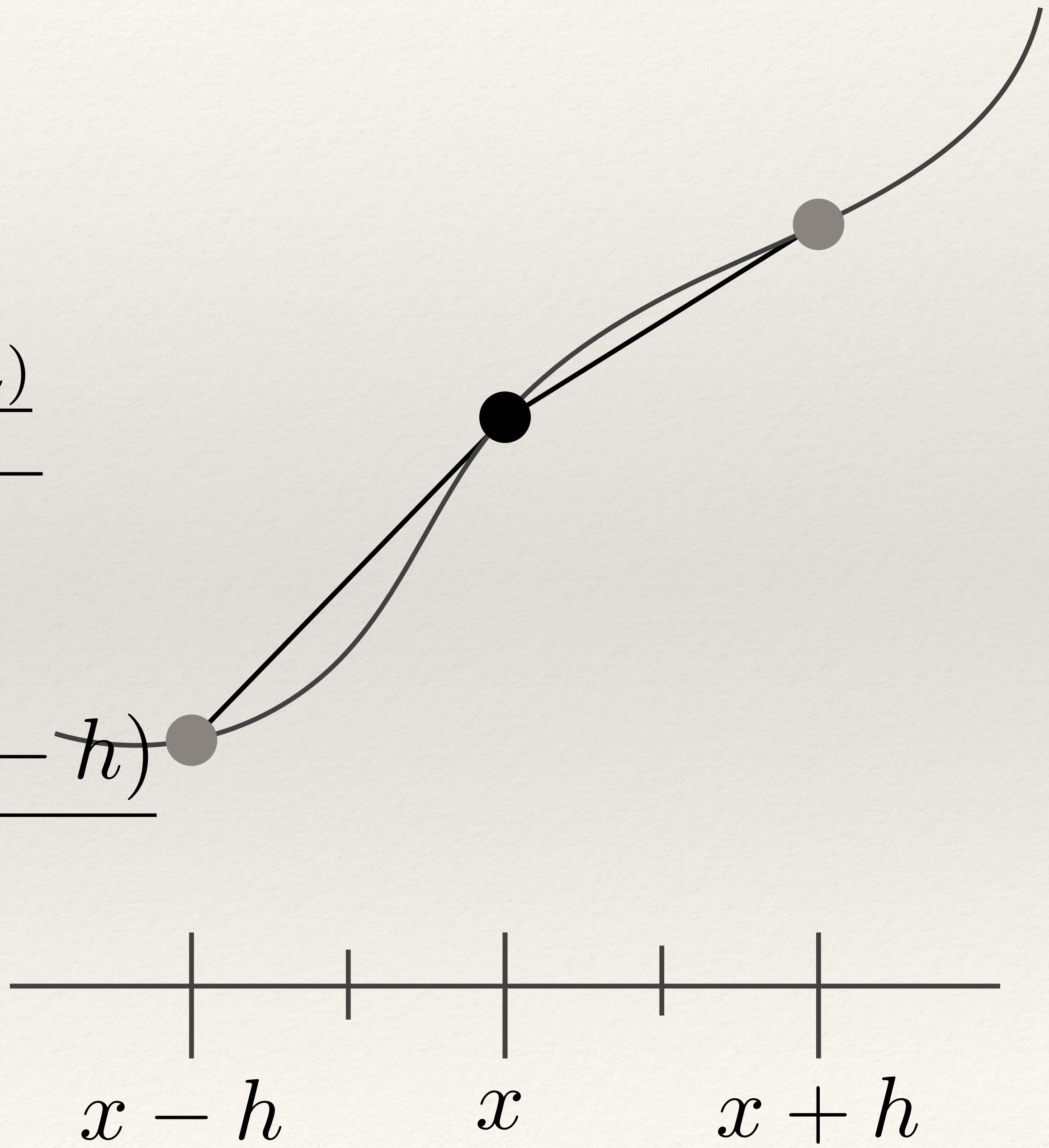


Higher Derivatives

$$f''(x) \approx \frac{f'(x+\frac{h}{2}) - f'(x-\frac{h}{2})}{h}$$

$$\approx \frac{\frac{f(x+h)-f(x)}{h} - \frac{f(x)-f(x-h)}{h}}{h}$$

$$\approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

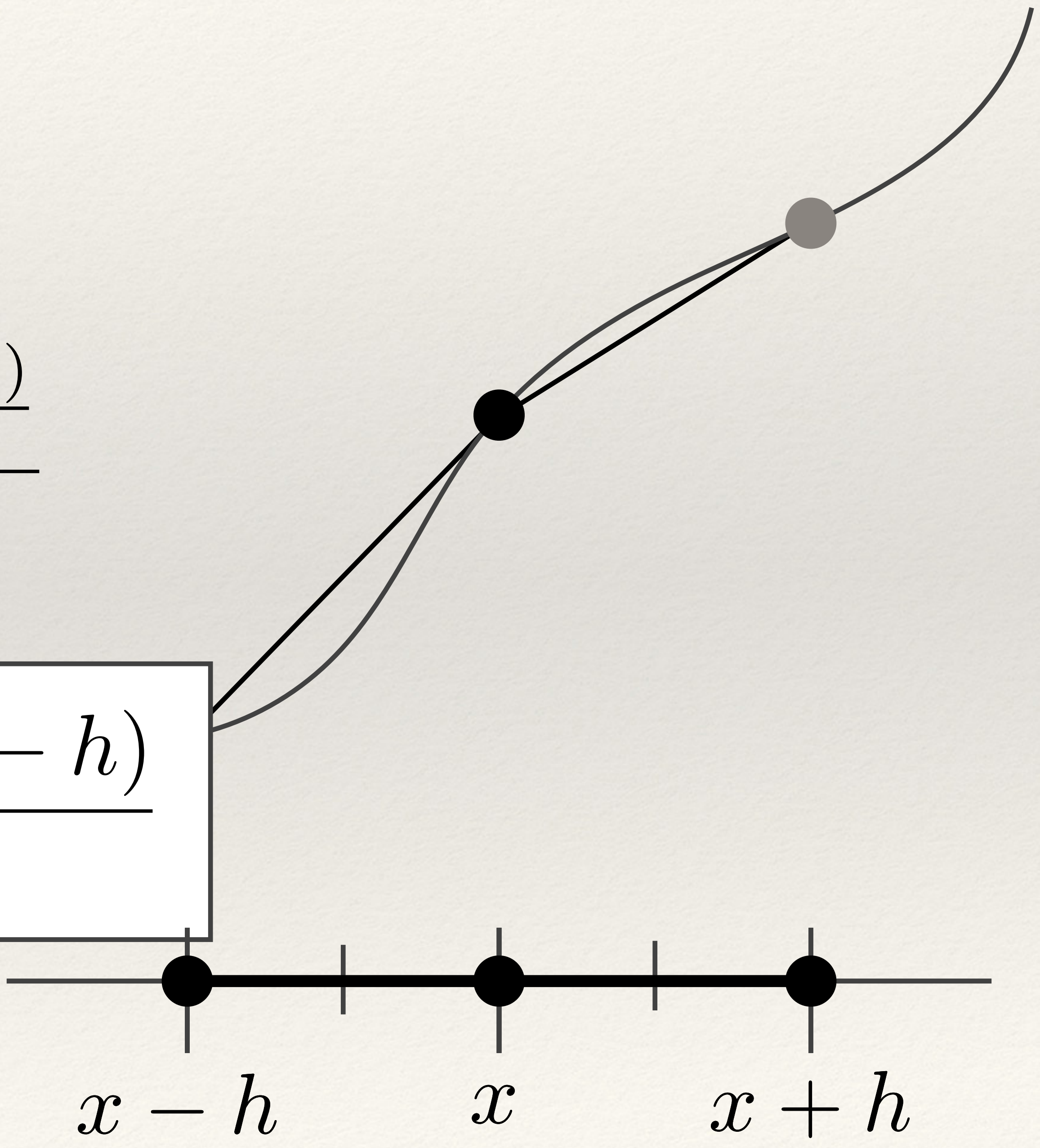


Higher Derivatives

$$f''(x) \approx \frac{f'(x+\frac{h}{2}) - f'(x-\frac{h}{2})}{h}$$

$$\approx \frac{\frac{f(x+h)-f(x)}{h} - \frac{f(x)-f(x-h)}{h}}{h}$$

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$



3 - point stencil

Laplacian Operator

$$\rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \mu \Delta \mathbf{u} + \mathbf{f}$$

$$\nabla \cdot \mathbf{u} = 0$$

- In 2D $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

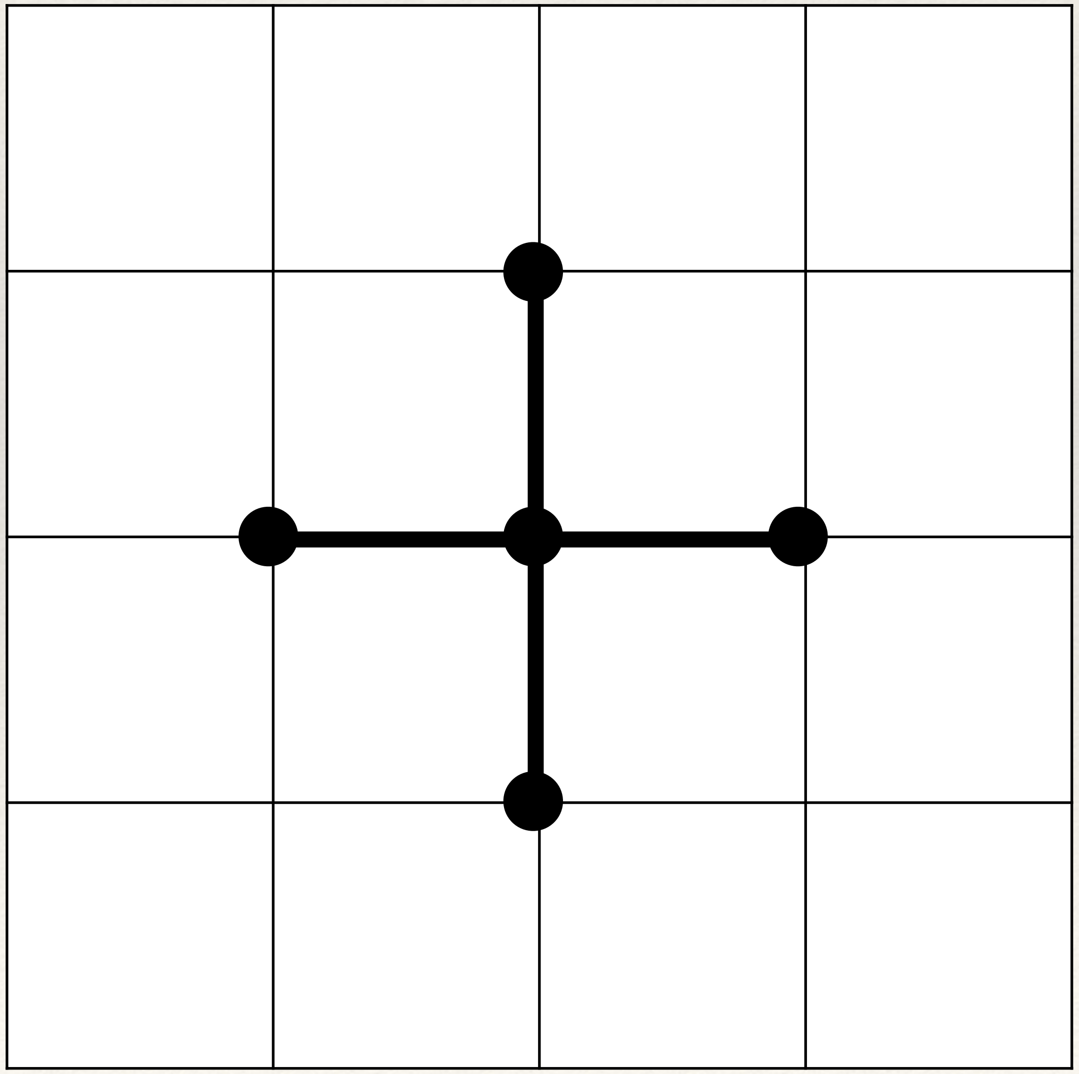
$$\frac{\partial^2 u(x,y)}{\partial x^2} \approx \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2}$$

$$\frac{\partial^2 u(x,y)}{\partial y^2} \approx \frac{u(x,y+h) - 2u(x,y) + u(x,y-h)}{h^2}$$

Laplacian Operator

$$\Delta u \approx \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}}{h^2}$$

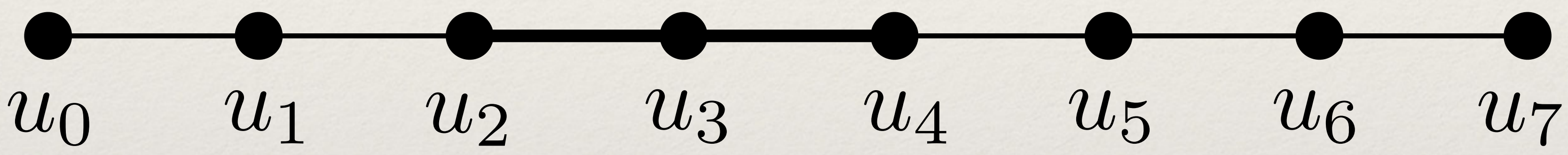
5 - point stencil



Poisson Equation (1D)

$$u_{xx} = f, \quad x \in \Omega$$

$$u(x) = \bar{u}(x), \quad x \in \partial\Omega$$



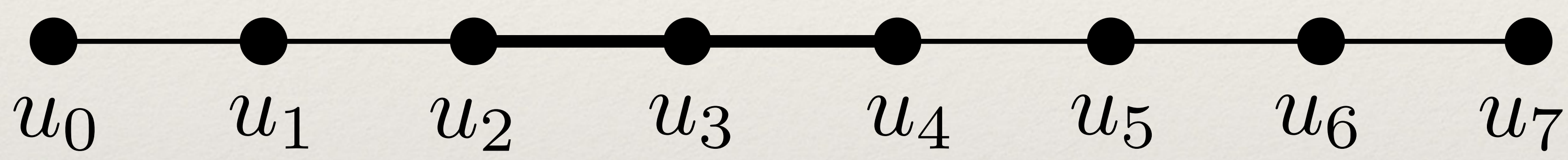
- At each interior node $\quad \dfrac{u_{i-1} - 2u_i + u_{i+1}}{h^2} = f_i$

- Boundary nodes $\quad u_0 = \bar{u}(x_0), u_7 = \bar{u}(x_7)$

Poisson Equation (1D)

$$u_{xx} = f, \quad x \in \Omega$$

$$u(x) = \bar{u}(x), \quad x \in \partial\Omega$$



linear system

$$\frac{1}{h^2}\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix} = \begin{pmatrix} f_1 - \frac{\bar{u}(x_0)}{h^2} \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 - \frac{\bar{u}(x_7)}{h^2} \end{pmatrix}$$

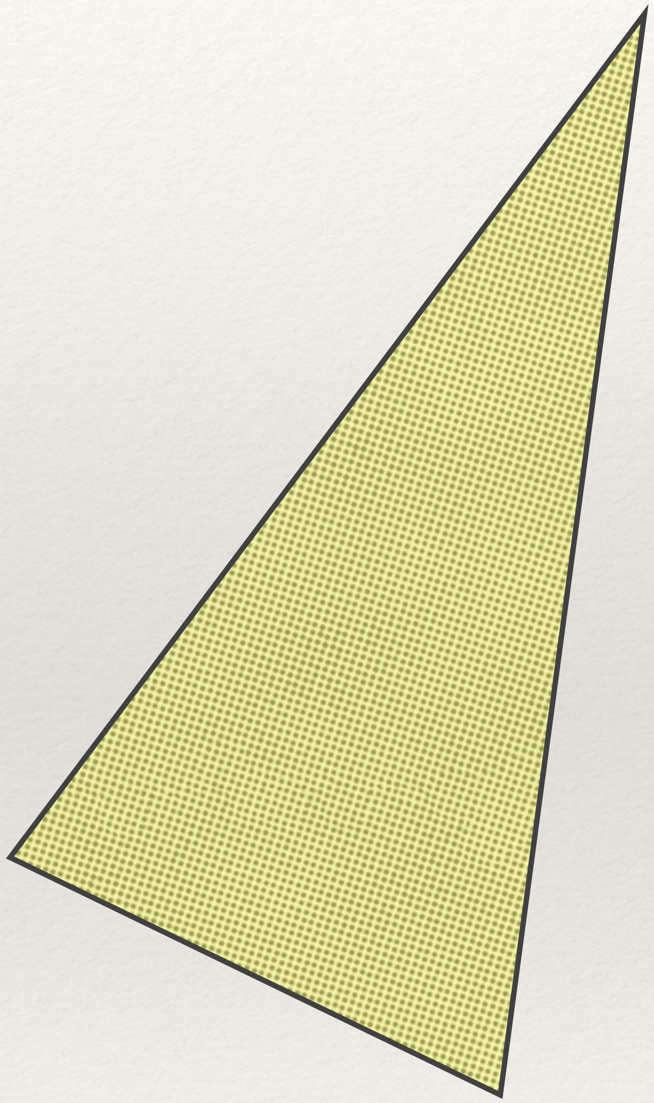
Finite Elements

Finite Elements

- Discretize the space of representable functions, instead of discretizing derivatives
- Method:
  - Discretize space into a finite set of elements
  - Choose a set of basis functions over the elements (e.g. piecewise linear)
  - *Galerkin* projection onto these basis functions
  - Solve the problem

Lets try an example:
Linear Finite Elements for Elasticity

Choose Element Type: Triangle



Choose Basis Functions: Piecewise Linear

Project Deformation Function onto the Piecewise Linear Function Space

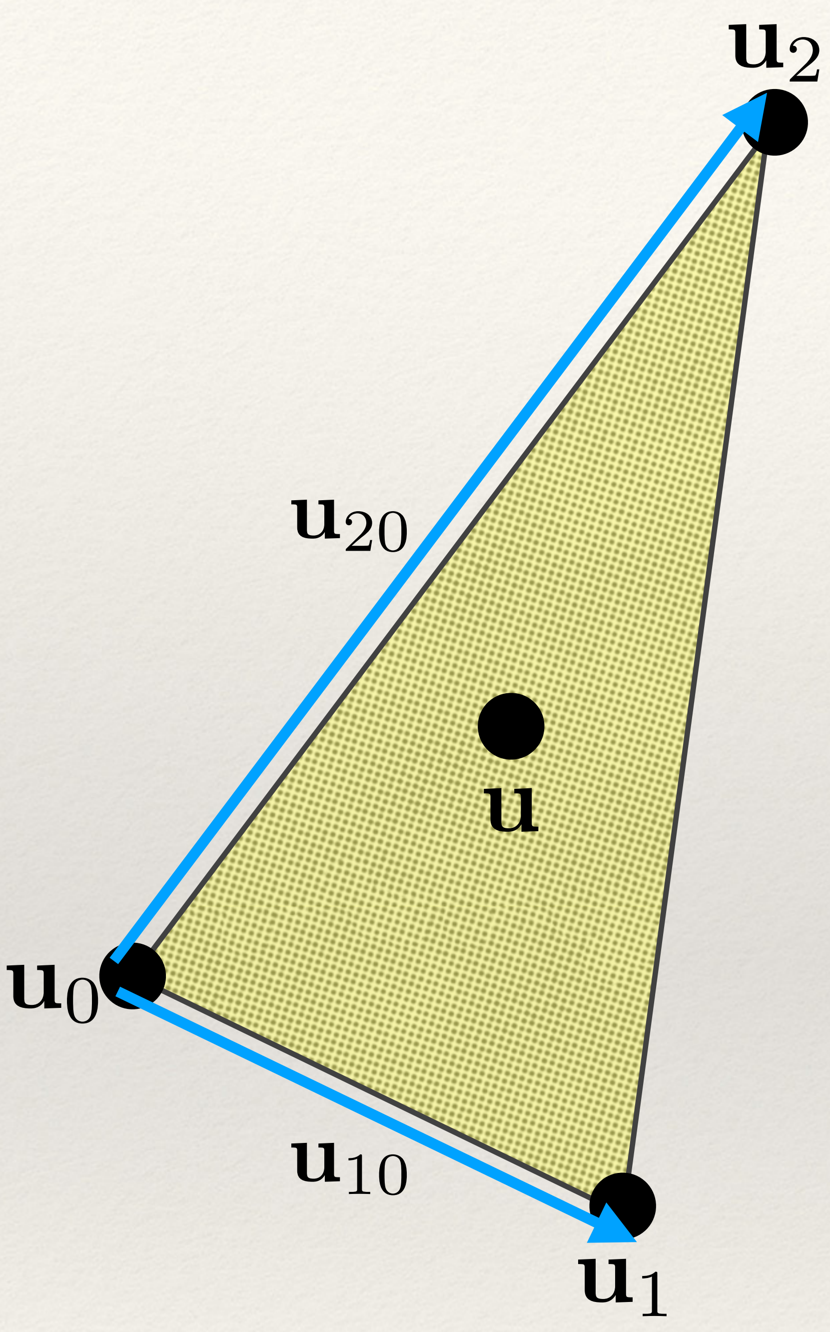
$$\mathbf{x}(\mathbf{u}) = \mathbf{x}(\mathbf{u}_0) + \mathbf{F}\left(\mathbf{u} - \mathbf{u}_0\right)$$

Solve the Problem

We know we can compute forces from $\mathbf{F}$

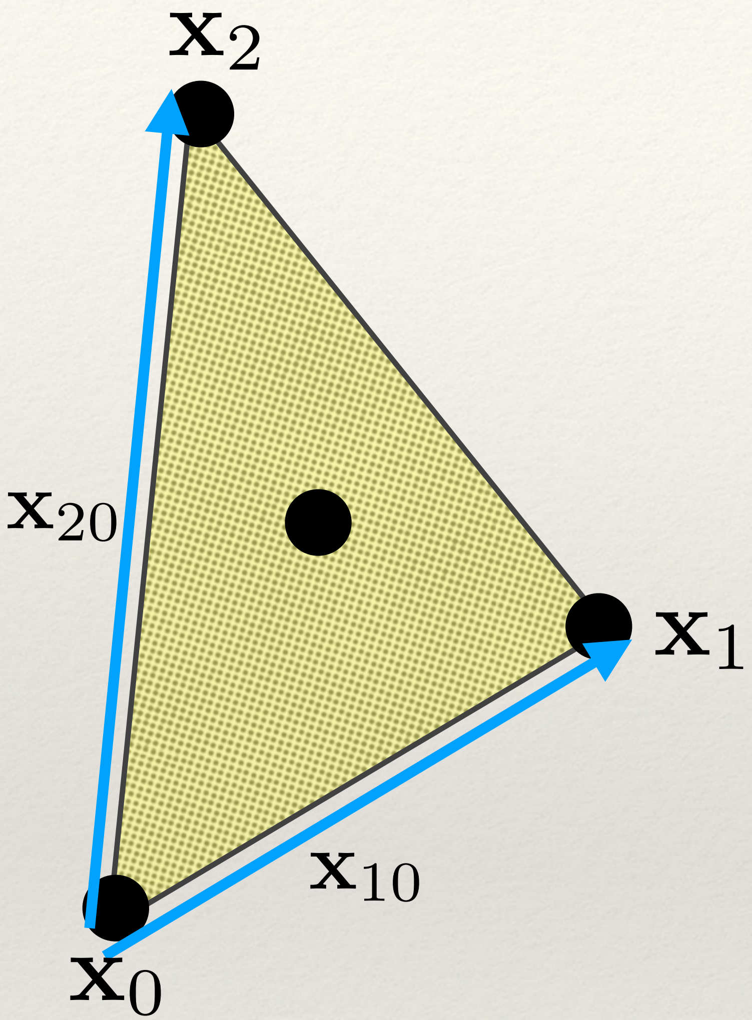
But, how do we compute $\mathbf{F}$?

$$\left(\text{in } \mathbf{x}(\mathbf{u}) = \mathbf{x}(\mathbf{u}_0) + \mathbf{F}\,(\mathbf{u} - \mathbf{u}_0) \right)$$



$$\mathbf{u} = \mathbf{u}_0 + \alpha\left(\mathbf{u}_1 - \mathbf{u}_0\right) + \beta\left(\mathbf{u}_2 - \mathbf{u}_0\right)$$

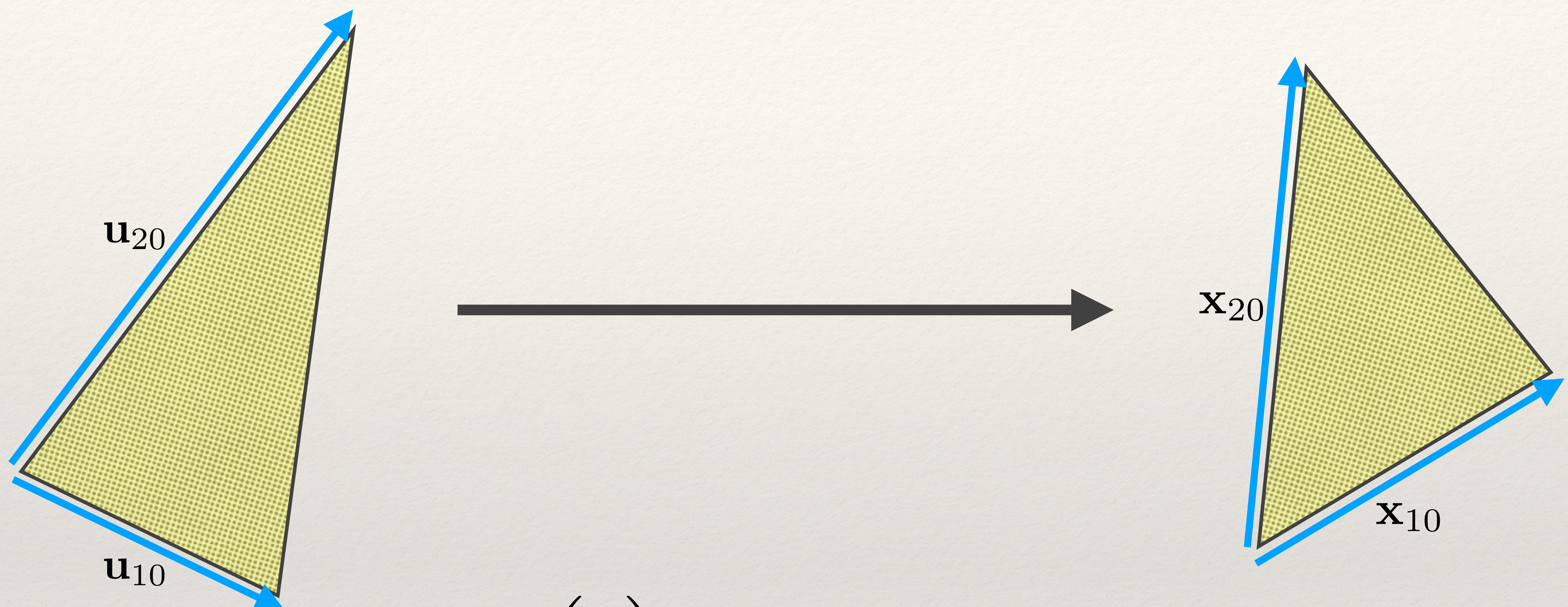
$$\mathbf{u} = \mathbf{u}_0 + \begin{pmatrix}\mathbf{u}_{10} & \mathbf{u}_{20}\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$$



$$\mathbf{x} = \mathbf{x}_0 + \alpha\left(\mathbf{x}_1 - \mathbf{x}_0\right) + \beta\left(\mathbf{x}_2 - \mathbf{x}_0\right)$$

$$\mathbf{x} = \mathbf{x}_0 + \begin{pmatrix}\mathbf{x}_{10} & \mathbf{x}_{20}\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$$

Rest/Material World

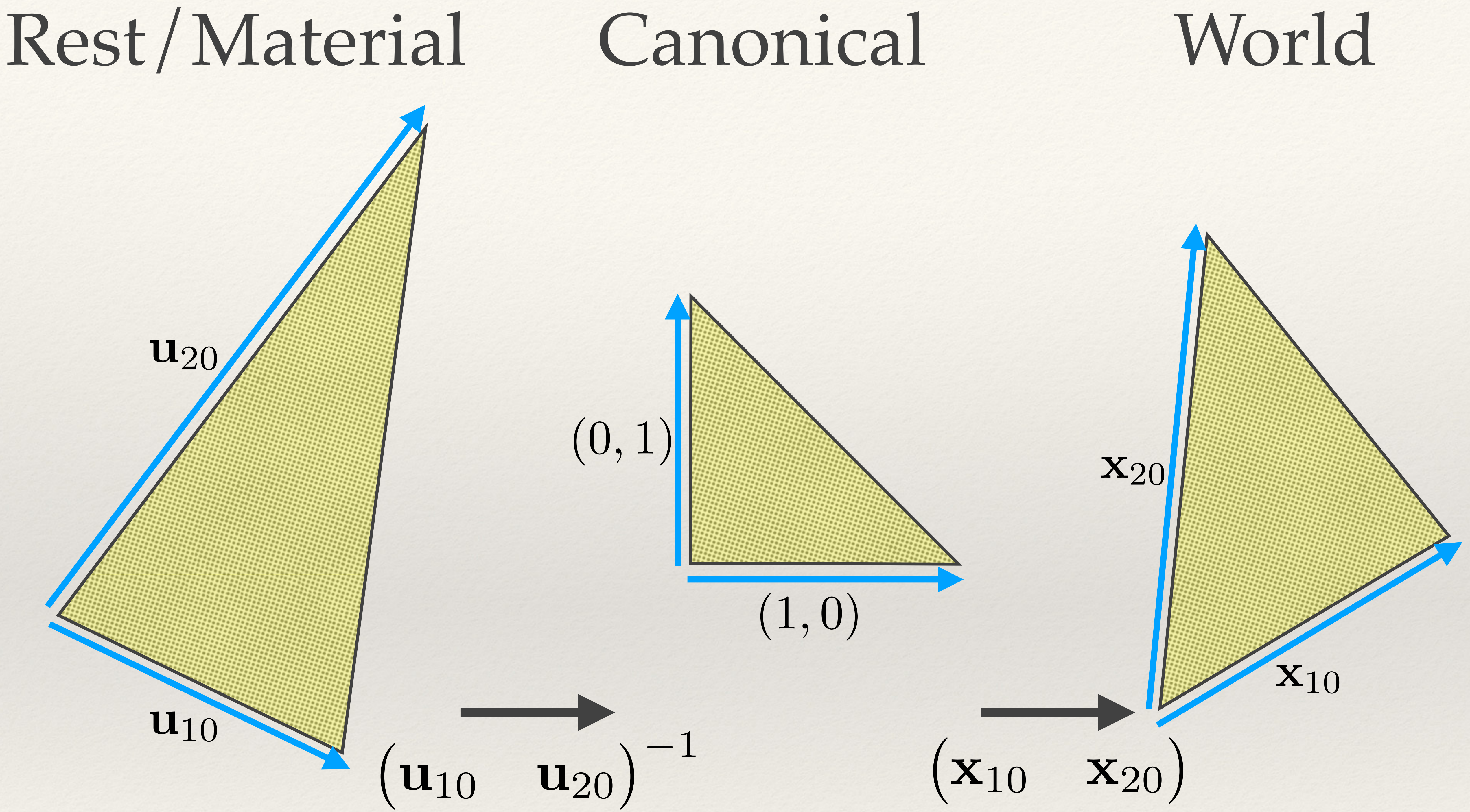


$$\mathbf{u} = \mathbf{u}_0 + \begin{pmatrix} \mathbf{u}_{10} & \mathbf{u}_{20} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\mathbf{x} = \mathbf{x}_0 + \begin{pmatrix} \mathbf{x}_{10} & \mathbf{x}_{20} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\mathbf{x}(\mathbf{u}) = \mathbf{x}_0 + \begin{pmatrix} \mathbf{x}_{10} & \mathbf{x}_{20} \end{pmatrix} \begin{pmatrix} \mathbf{u}_{10} & \mathbf{u}_{20} \end{pmatrix}^{-1} (\mathbf{u} - \mathbf{u}_0)$$

$$\mathbf{F} = \begin{pmatrix} \mathbf{x}_{10} & \mathbf{x}_{20} \end{pmatrix} \begin{pmatrix} \mathbf{u}_{10} & \mathbf{u}_{20} \end{pmatrix}^{-1}$$



$$\mathbf{F} = \begin{pmatrix}\mathbf{x}_{10} & \mathbf{x}_{20}\end{pmatrix} \begin{pmatrix}\mathbf{u}_{10} & \mathbf{u}_{20}\end{pmatrix}^{-1}$$

Solve the Problem

$$\epsilon = \frac{1}{2}\left(\tilde{\mathbf{F}}^T + \tilde{\mathbf{F}}\right) - \mathbf{I}$$

where $\mathbf{F} = \mathbf{Q}\tilde{\mathbf{F}}$

$$\sigma = \lambda \mathrm{Tr}\left(\epsilon\right)\mathbf{I} + 2\mu\epsilon$$

$$\mathbf{f} = \mathbf{Q}\sigma\mathbf{n}_i$$

where $\mathbf{n}_i$
are the normals of the opposite faces
in rest space

IV. Temporal Integration

Explicit Integration

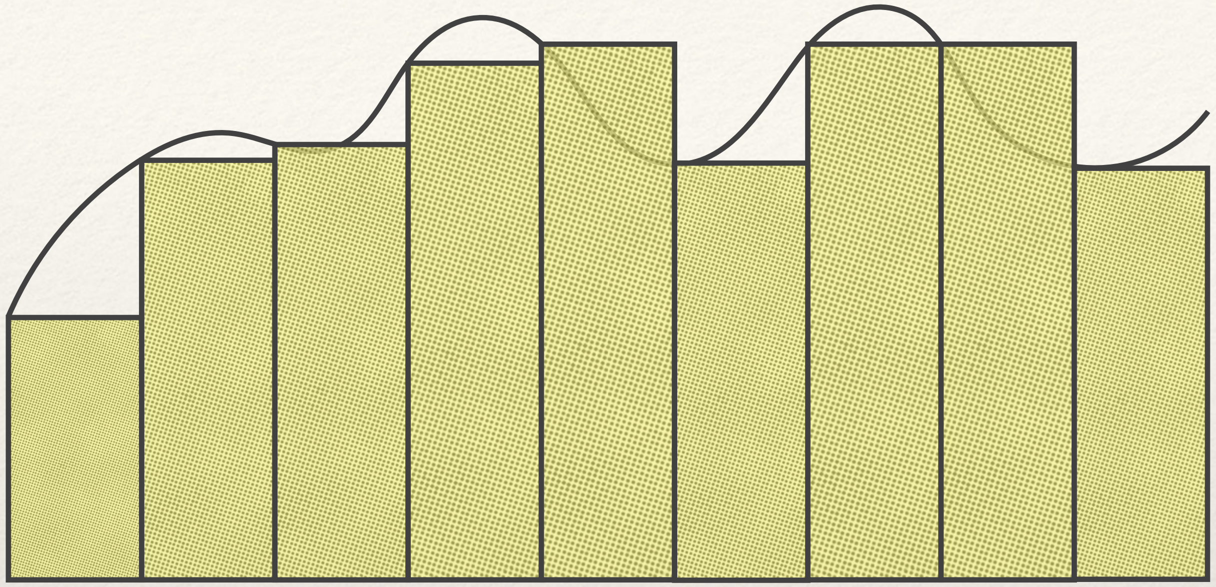
Explicit formula for (t+1)
in terms of quantities known at time t

$$\mathbf{x}_p(t + \Delta t) = \mathbf{x}_p(t) + \Delta t \cdot \mathbf{v}(\mathbf{x}_p, t)$$

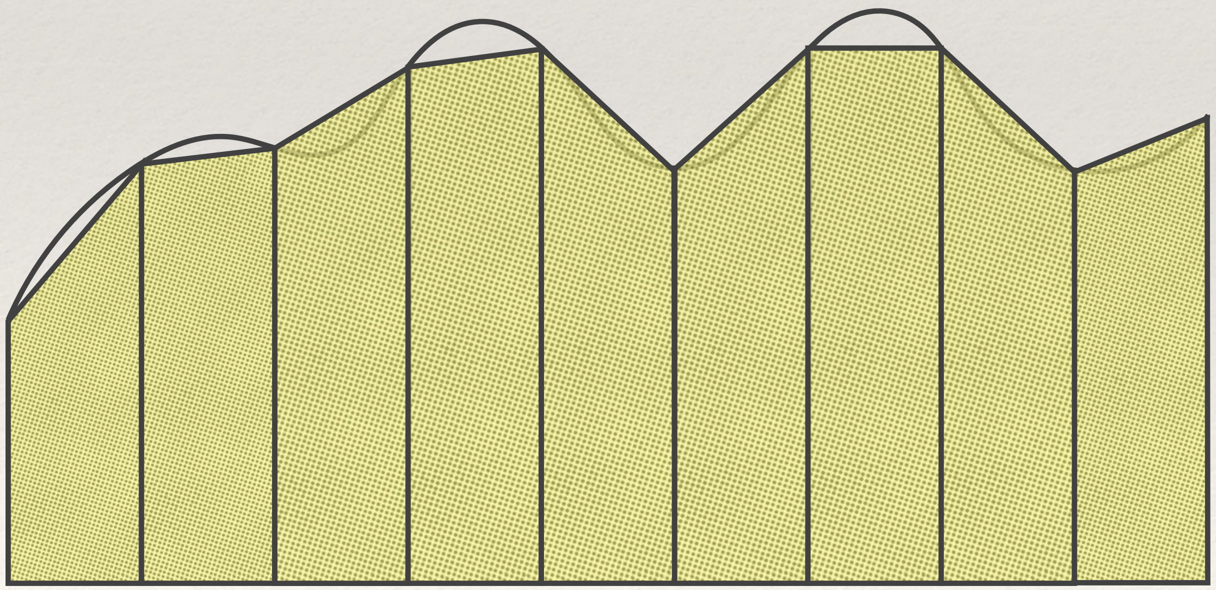
Note: everything on the right hand side
is evaluated at time t

Choose Your Integration Scheme Wisely

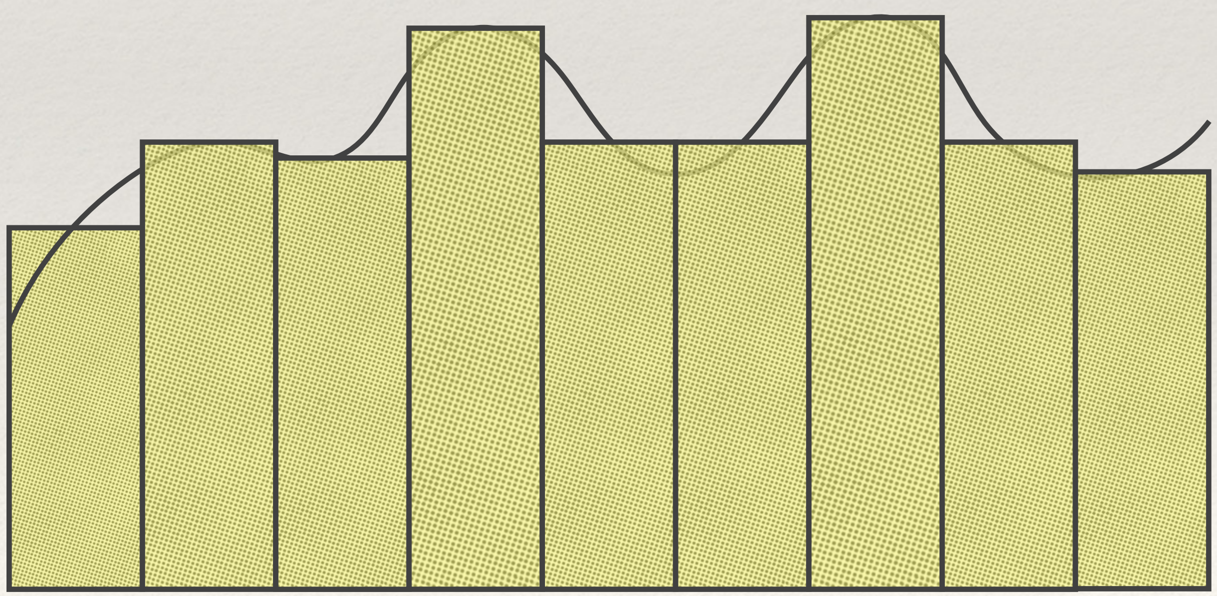
Trapezoidal Rule vs. Midpoint Method



Forward Euler



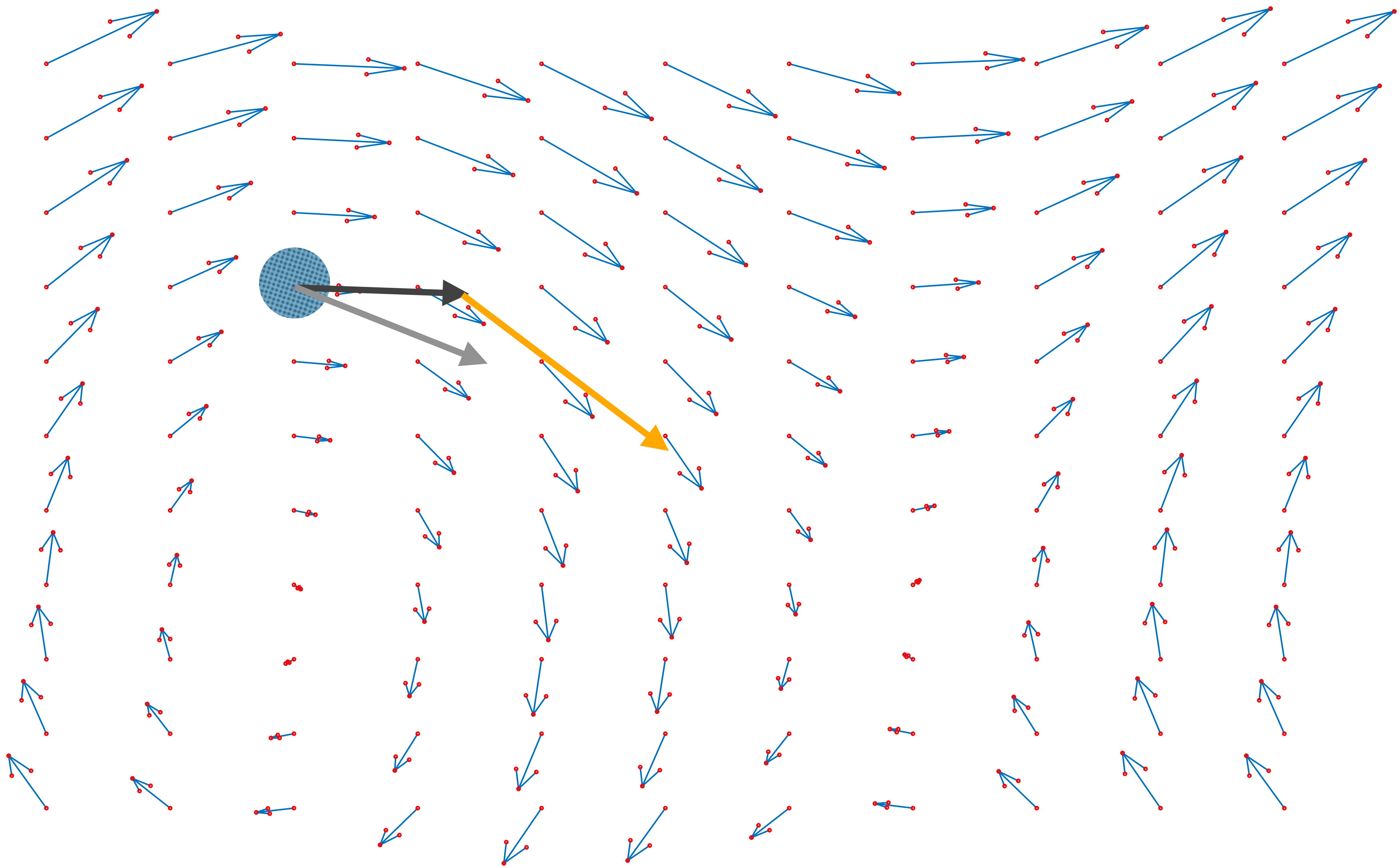
Trapezoidal Rule



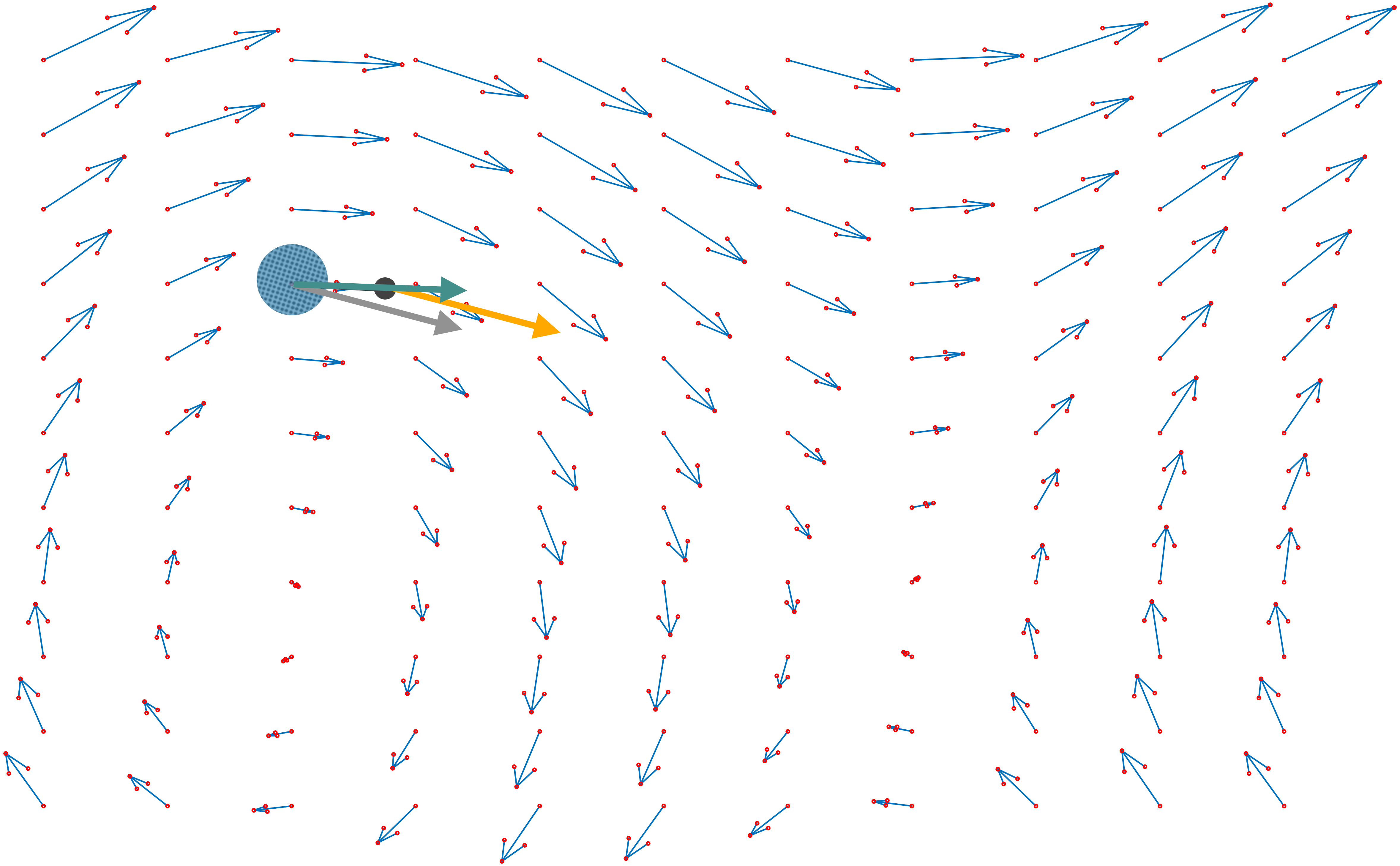
Midpoint Method

Trapezoidal Rule

Average



Midpoint Method



Trapezoidal Rule vs. Midpoint Method

- Both second-order Runge-Kutta methods (same accuracy)
- Very different behavior
  - Trapezoidal rule is smoother, more damped looking
  - Midpoint Method keeps more energy, but can be noisy/aliased

Position Updates for

$$\frac{d^2\mathbf{x}_p(t)}{dt^2} = \frac{\mathbf{f}(\mathbf{x}_p, t)}{m_p}$$

Three Position Updates

$$\mathbf{x}_p(t + \Delta t) = \mathbf{x}_p(t) + dt \cdot \mathbf{v}_p(t)$$

$$\mathbf{x}_p(t + \Delta t) = \mathbf{x}_p(t) + \frac{dt}{2} \cdot (\mathbf{v}_p(t) + \mathbf{v}_p(t + \Delta t))$$

$$\mathbf{x}_p(t + \Delta t) = \mathbf{x}_p(t) + dt \cdot \mathbf{v}_p(t + \Delta t)$$

"Stiff" Problems

Consider the IVP:

$$\mathbf{x}(0) = \mathbf{x}_0 \qquad \mathbf{v}(0) = 0 \qquad \mathbf{f} = -k\mathbf{x}$$

After one time step:

$$\mathbf{x}(\Delta t) = \left(1 - \frac{\Delta t^2 k}{m}\right)\mathbf{x}_0$$

If $\Delta t > \sqrt{\frac{2m}{k}}$

the spring will be more extended then when we started

If We Want to Take Bigger Timesteps

Implicit Integration

Replace:

$$\mathbf{v}(t+\Delta t) = \mathbf{v}(t) + \Delta t \cdot \mathbf{M}^{-1}\mathbf{f}(\mathbf{x}(t), t)$$

$$\mathbf{x}(t+\Delta t) = \mathbf{x}(t) + \Delta t \cdot \mathbf{v}(t+\Delta t)$$

With:

$$\mathbf{v}(t+\Delta t) = \mathbf{v}(t) + \Delta t \cdot \mathbf{M}^{-1}\mathbf{f}(\mathbf{x}(t+\Delta t), t+\Delta t)$$

$$\mathbf{x}(t+\Delta t) = \mathbf{x}(t) + \Delta t \cdot \mathbf{v}(t+\Delta t)$$

Applied to Soft Bodies

$$\mathbf{K}(\mathbf{x} - \mathbf{x}_0) + \mathbf{D}(\mathbf{v}) + \mathbf{M}\mathbf{a} = \mathbf{f}_{ext}$$

Linearize

$$\mathbf{K}\mathbf{x} - \mathbf{K}\mathbf{x}_0 + \mathbf{D}\mathbf{v} + \mathbf{M}\mathbf{a} = \mathbf{f}_{ext}$$

$$+$$

$$\mathbf{x}(t + \Delta t) = \mathbf{x}(t) + \Delta t \cdot \mathbf{v}(t + \Delta t)$$

substitute into

$$\mathbf{v}(t + \Delta t) = \mathbf{v}(t) + \Delta t \cdot \mathbf{M}^{-1}\mathbf{f}(\mathbf{x}(t + \Delta t), t + \Delta t)$$

re-arrange

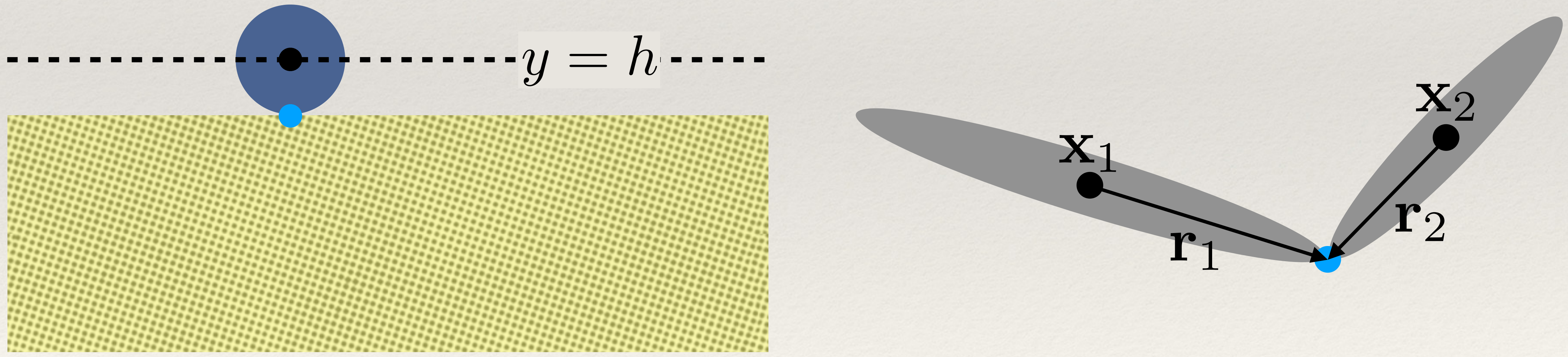
$$\left(\mathbf{M} + \Delta t^2 \mathbf{K} + \Delta t \mathbf{D}\right) \mathbf{v}(t + \Delta t) = \mathbf{M}\mathbf{v}(t) + \Delta t \left(-\mathbf{K}\left(\mathbf{x}(t) - \mathbf{x}_0\right) + \mathbf{f}_{ext}\right)$$

$$\underbrace{\left(\mathbf{M} + \Delta t^2 \mathbf{K} + \Delta t \mathbf{D}\right)}_{\substack{\text{Linear System} \\ \text{Sparse, Symmetric}}} \mathbf{v}(t + \Delta t) = \underbrace{\mathbf{M}\mathbf{v}(t)}_{\text{Momentum}} + \overbrace{\Delta t \left(-\mathbf{K}\left(\mathbf{x}(t) - \mathbf{x}_0\right) + \mathbf{f}_{ext}\right)}^{\substack{\text{Euler step of elastic} \\ \text{and external forces}}}$$

V. Constraints

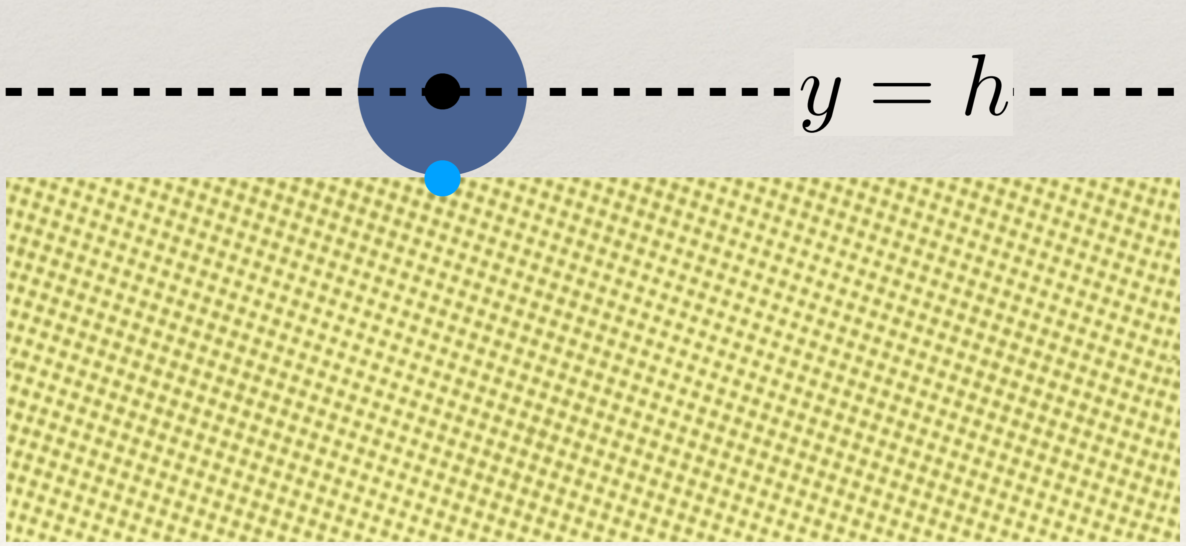
Constraints

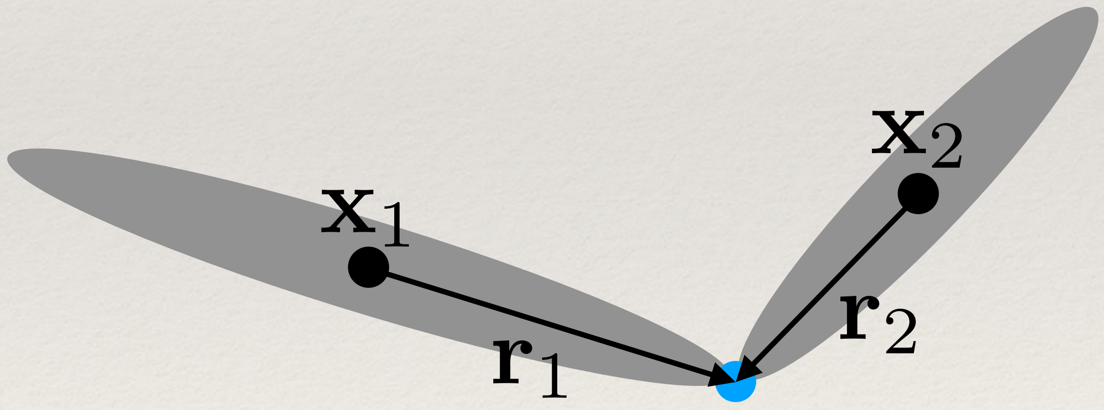
- Geometric relationships that must be satisfied



Constraints

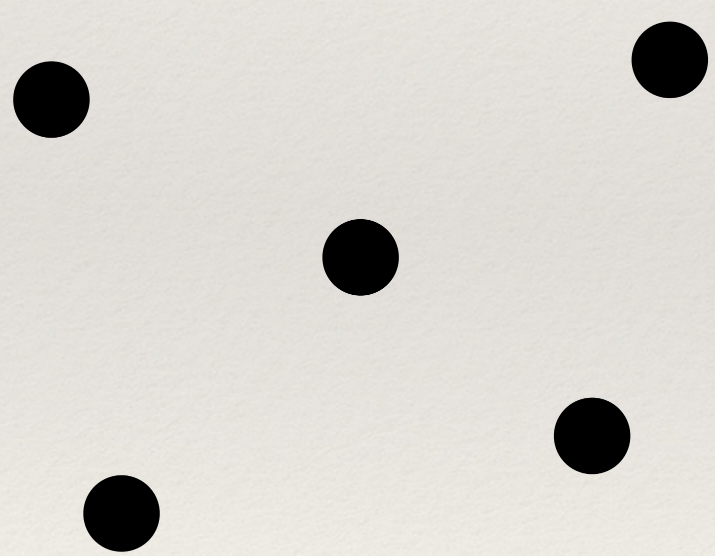
- Constraint forces arises in response to other forces to maintain the constraint





Degrees of Freedom (DoF)

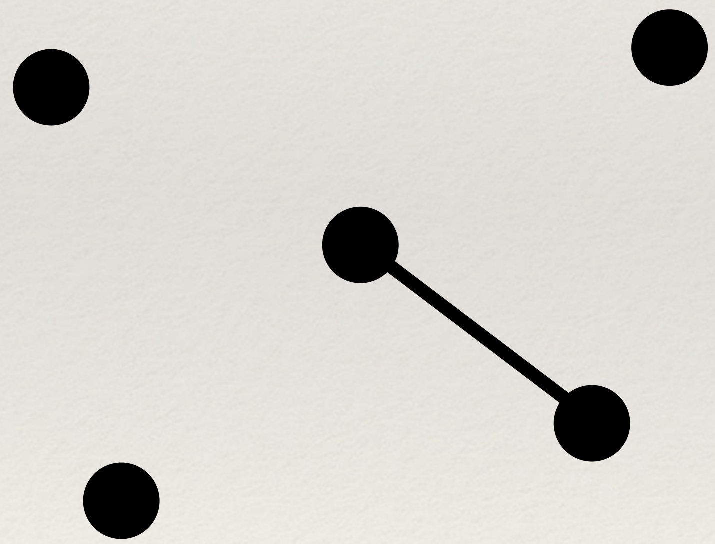
- Number of independent parameters describing configuration

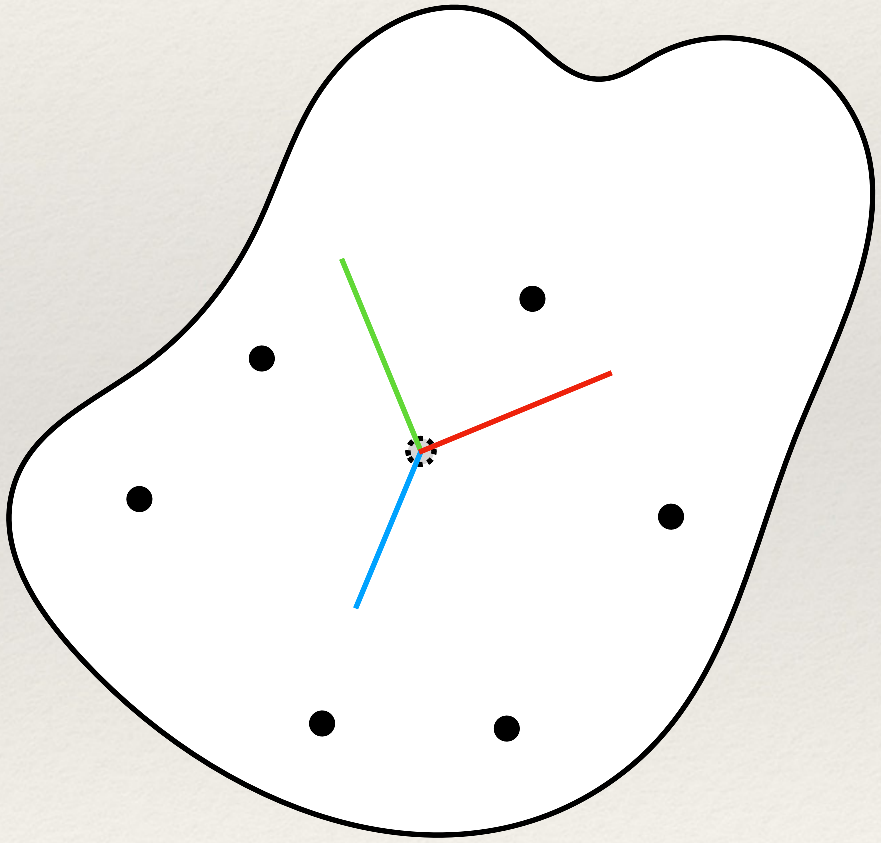


$3n$

Degrees of Freedom (DOF)

- Number of independent parameters describing configuration





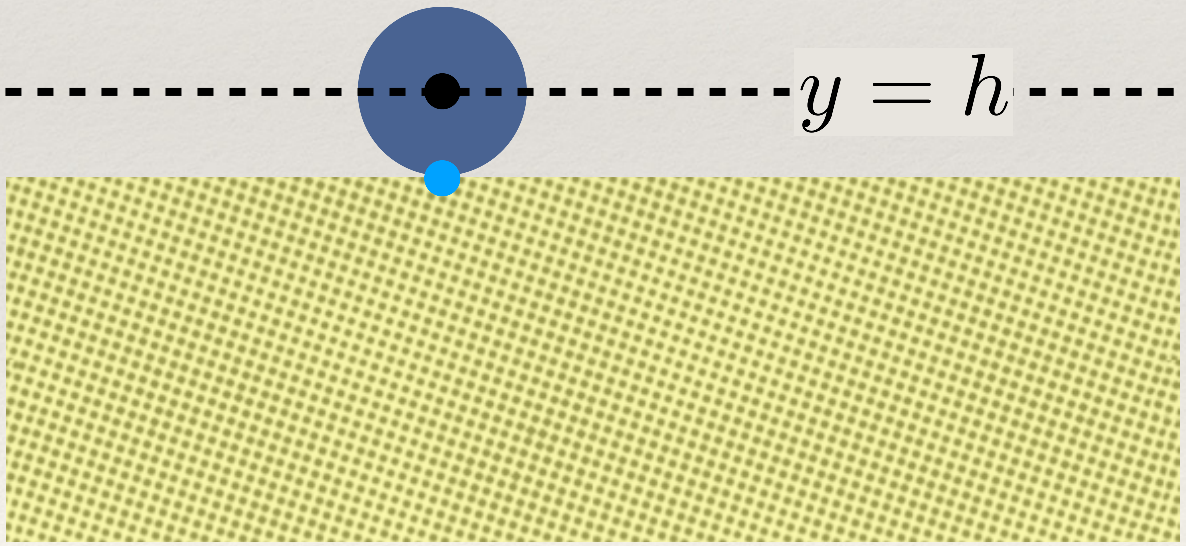
$3n - 1$

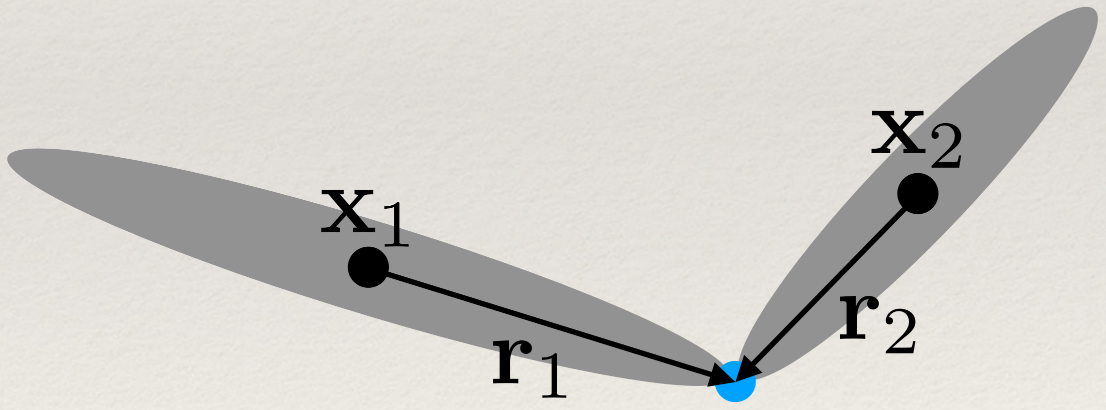
6

Unilateral/Bilateral Constraints

$$\mathbf{g}(\mathbf{x}, t) \geq \mathbf{0}$$

$$\mathbf{g}(\mathbf{x}, t) = \mathbf{0}$$





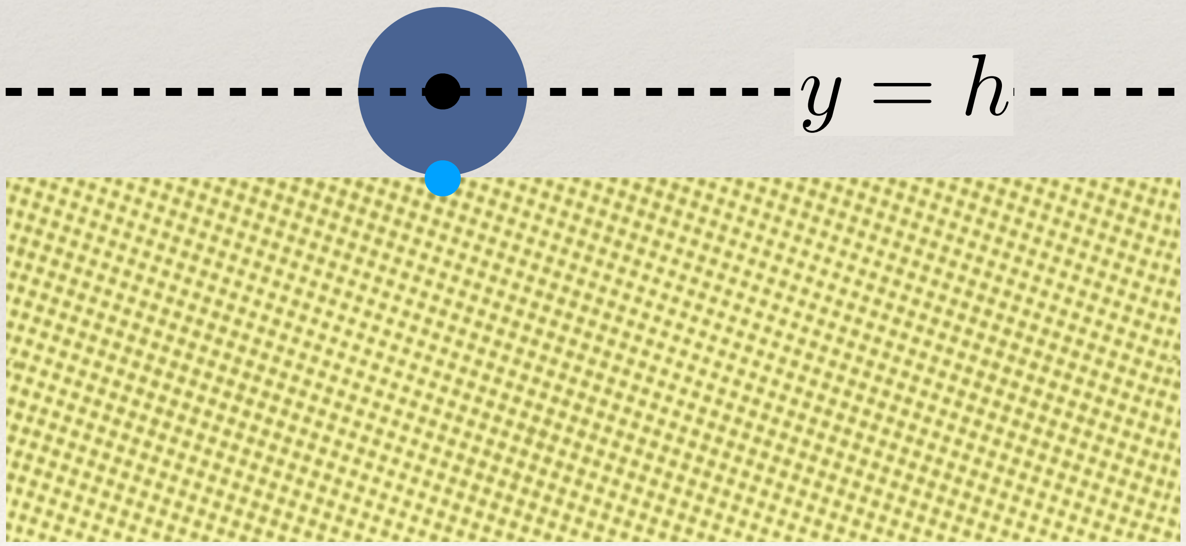
Unilateral/Bilateral Constraints

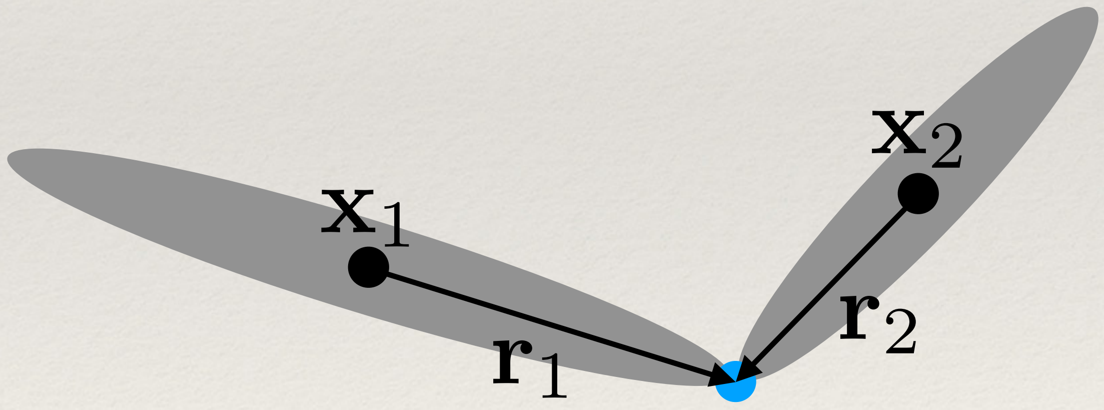
$$\mathbf{g}(\mathbf{x}, t) \geq \mathbf{0}$$

$$\mathbf{g}(\mathbf{x}, t) = \mathbf{0}$$

$$y_1 - h \geq 0$$

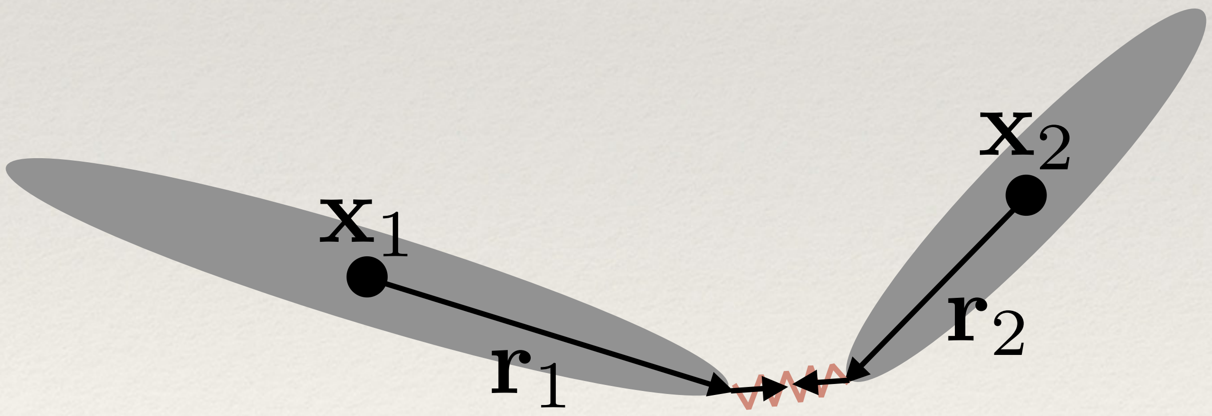
$$(\mathbf{x}_1 + \mathbf{r}_1) - (\mathbf{x}_2 + \mathbf{r}_2) = \mathbf{0}$$

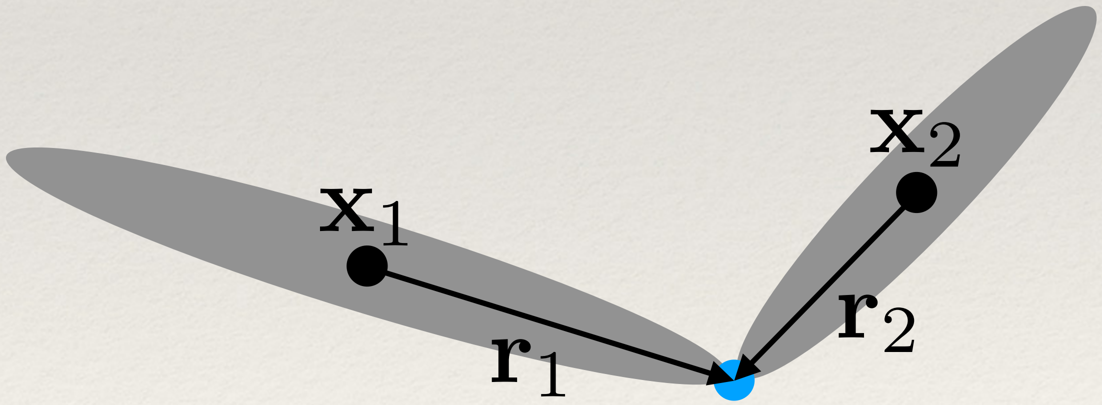




Soft vs. Hard Constraints

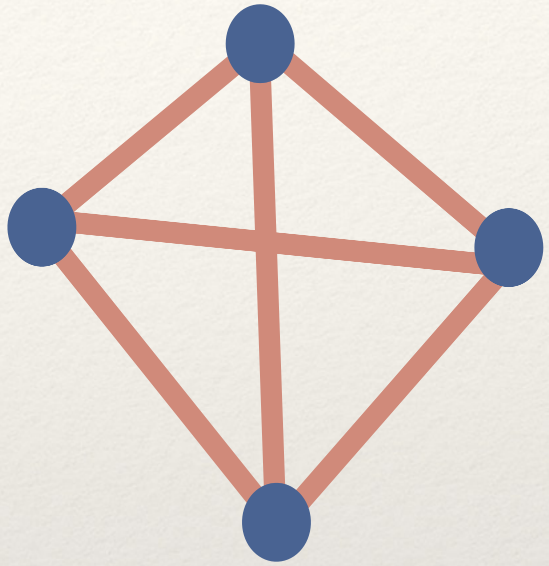
- **Soft constraint**: force competes with other forces in the system
- **Hard constraint**: force as strong as necessary to maintain constraint

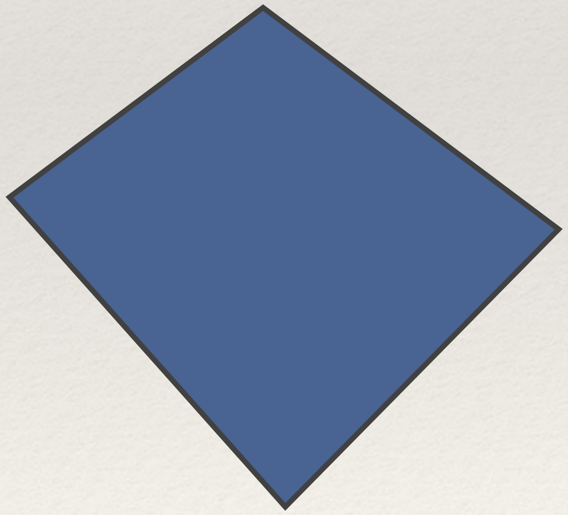




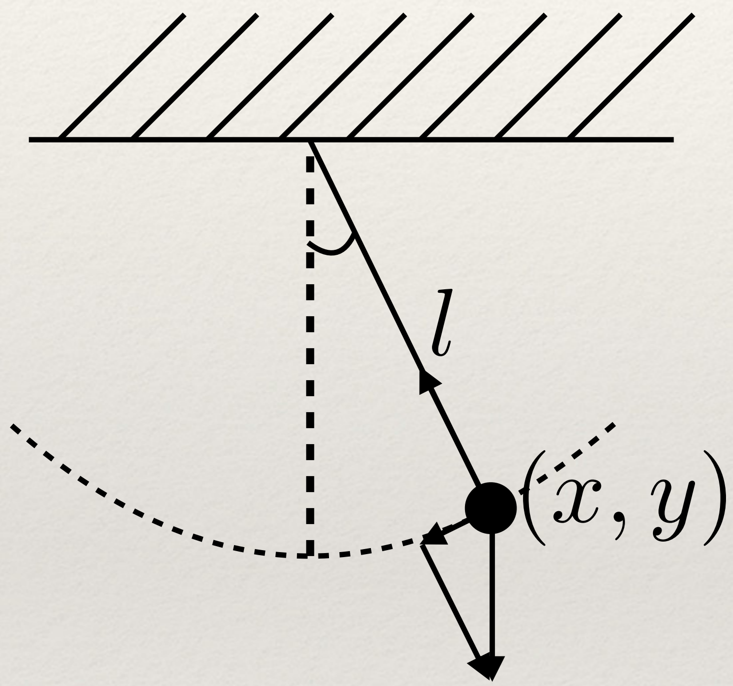
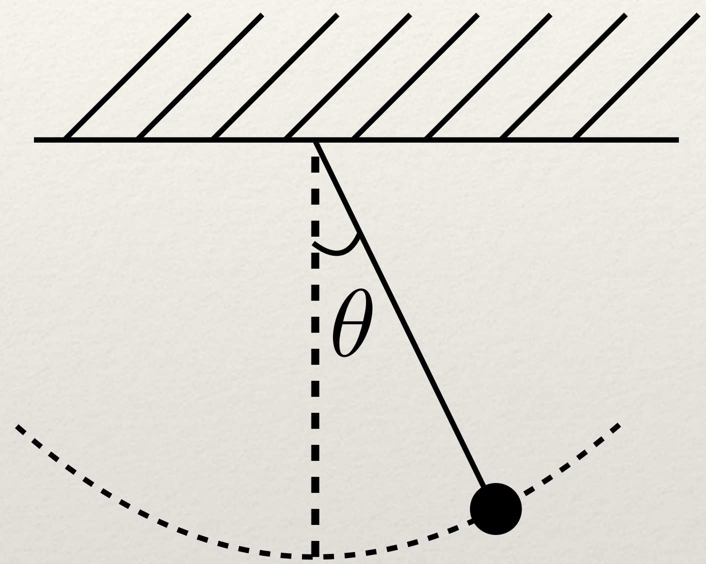
Constrained Dynamics: Solution Methods

- Maximal coordinates
  - constraint equation
  - forces to maintain constraint
- Generalized coordinates (a.k.a., reduced coordinates)
  - parameterize true DoF
  - no explicit constraint forces





Constrained Dynamics: Solution Methods

Maximal Coordinates	Generalized Coordinates
	
$m \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = -m\mathbf{g} + \mathbf{f}_c$ $g(x, y) = x^2 + y^2 - l^2 = 0$	$\ddot{\theta} + \frac{g}{l} \sin\theta = 0$

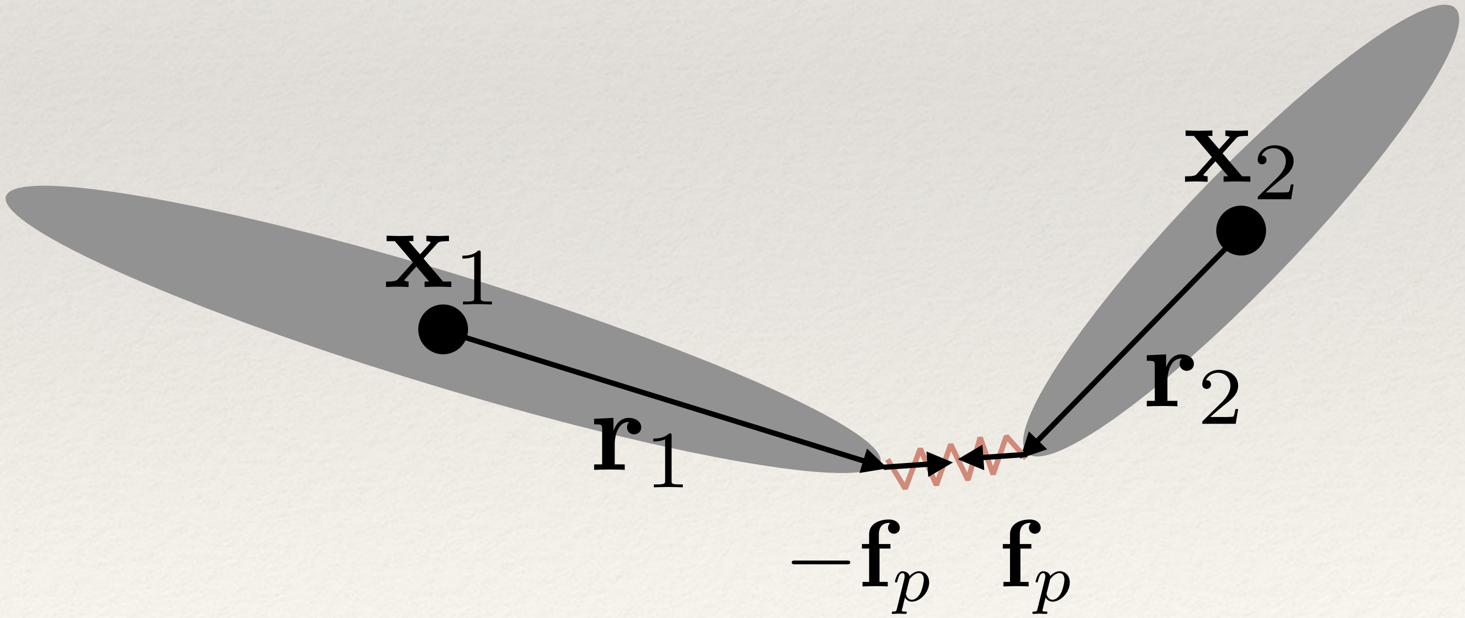
Penalty Methods

- Restoring force that acts to drive system to valid state

constraint: $(\mathbf{x}_1 + \mathbf{r}_1) - (\mathbf{x}_2 + \mathbf{r}_2) = \mathbf{0}$

penalty force: $\mathbf{f}_p = -k((\mathbf{x}_2 + \mathbf{r}_2) - (\mathbf{x}_1 + \mathbf{r}_1))$

$$m\mathbf{a} = \mathbf{f} + \mathbf{f}_p$$



Penalty Methods

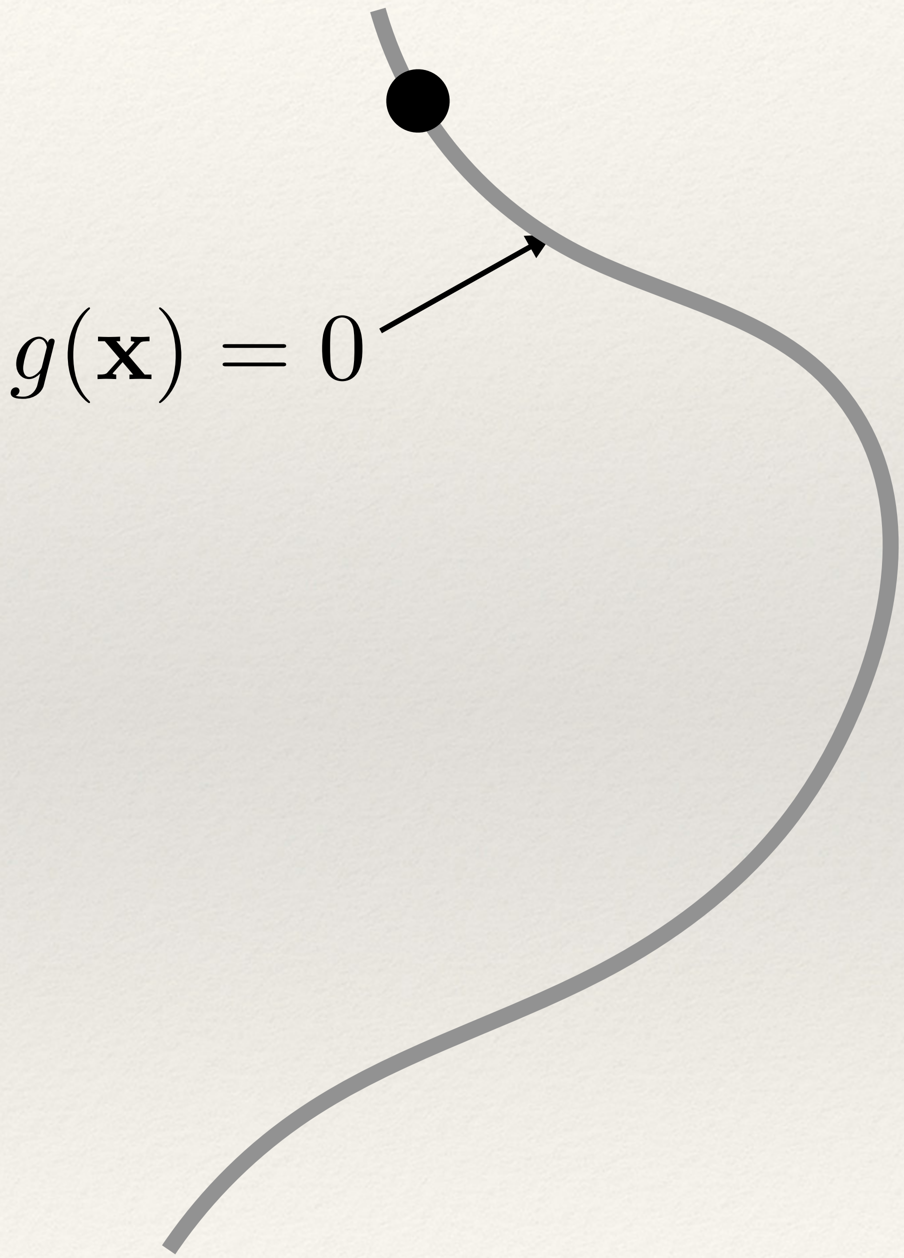
- ✓ Simple to add to solver
- – Must tune parameters (k)
- – Introduce stiff forces -> smaller time step or implicit integration
- – Constraint violation, oscillations

Lagrange Multiplier Methods

- Constraint force arises in response to other forces in the system
- Add unknowns to equations that represent strength of constraint force
- Add (differentiated) constraint equations

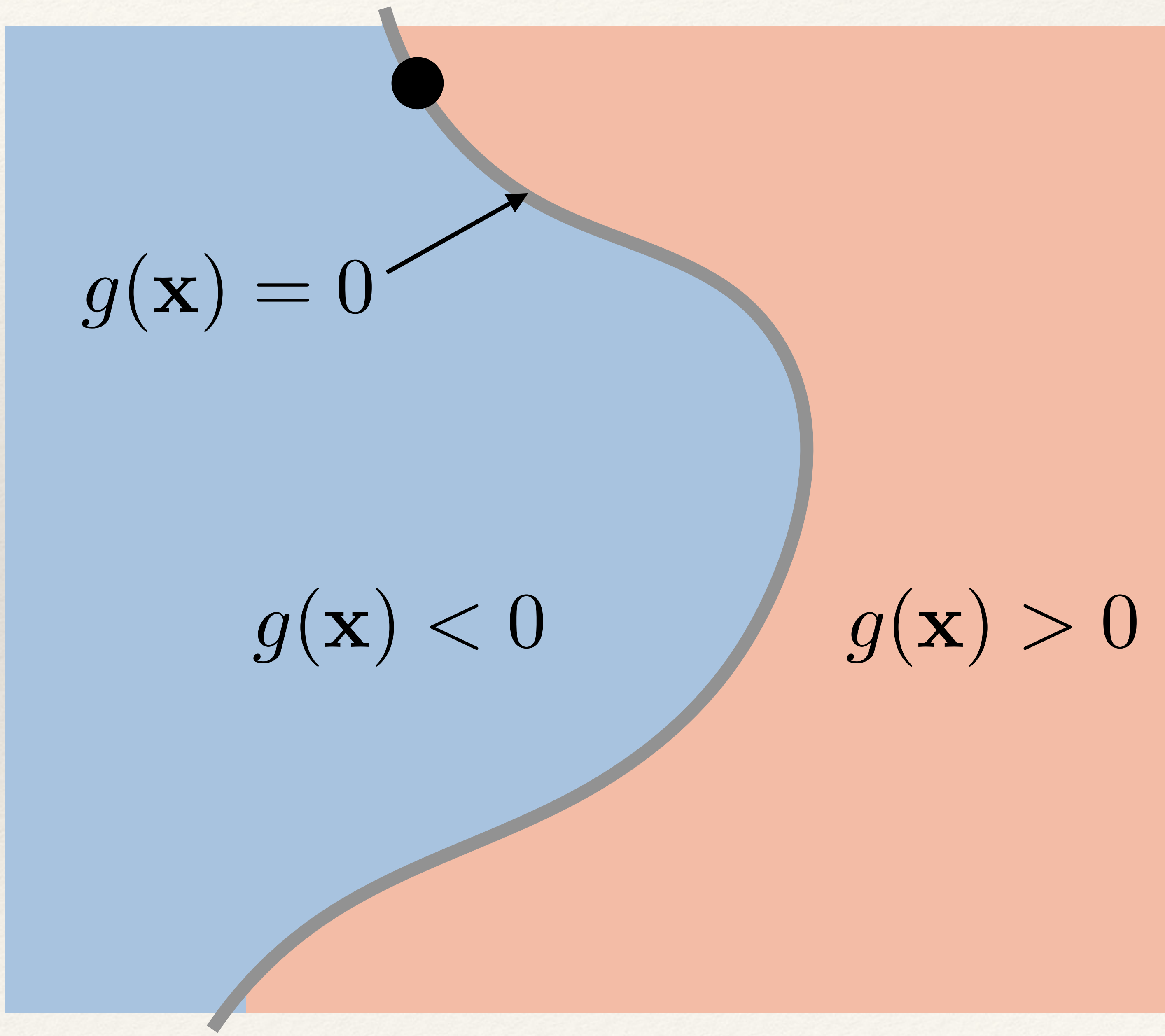
Constraint Implicit Surface

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)$$



Constraint Implicit Surface

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)$$

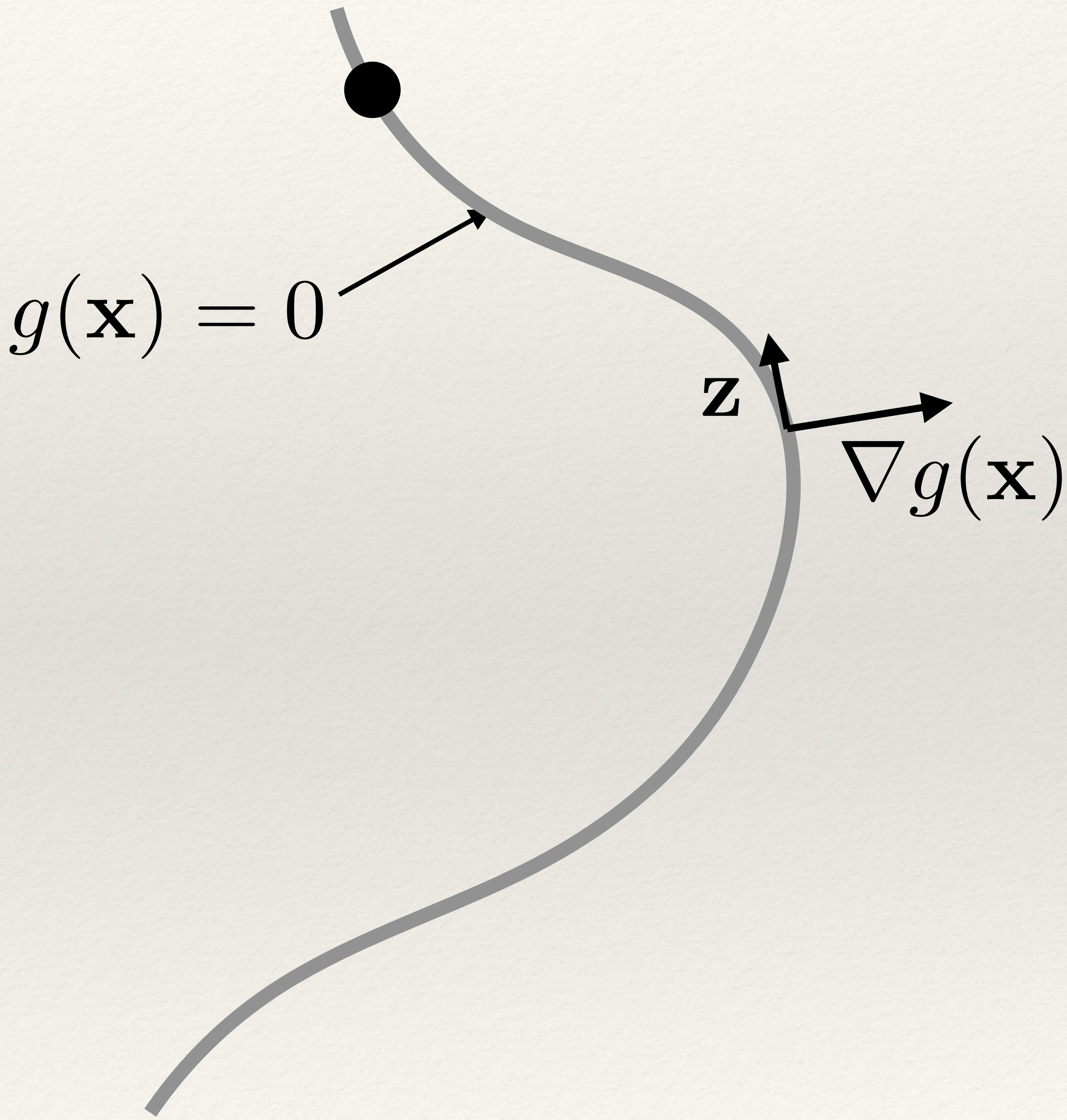


Constraint Implicit Surface

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)$$

- Gradient

$$\nabla g(\mathbf{x}) = \begin{pmatrix} \frac{\partial g}{\partial x_1}(\mathbf{x}) \\ \frac{\partial g}{\partial x_2}(\mathbf{x}) \\ \vdots \\ \frac{\partial g}{\partial x_n}(\mathbf{x}) \end{pmatrix}$$



Constraint Force

- Constraint force

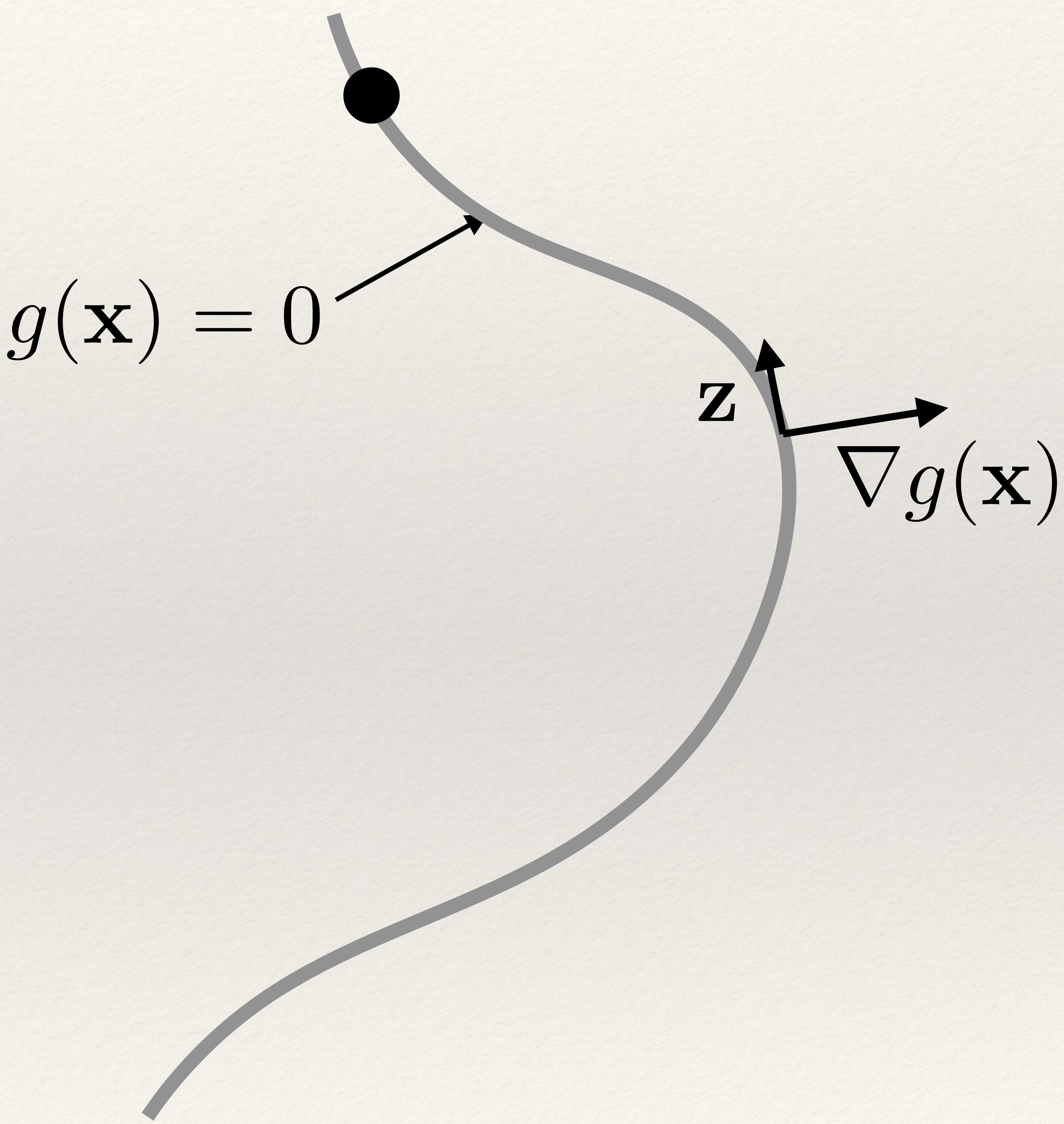
$$\mathbf{f}_c$$

- **Workless**

$$\mathbf{f}_c \cdot \mathbf{z} = 0$$

$$\Rightarrow \mathbf{f}_c = \lambda \nabla g(\mathbf{x})$$

Lagrange multiplier



Constraint Force

- Constraint force

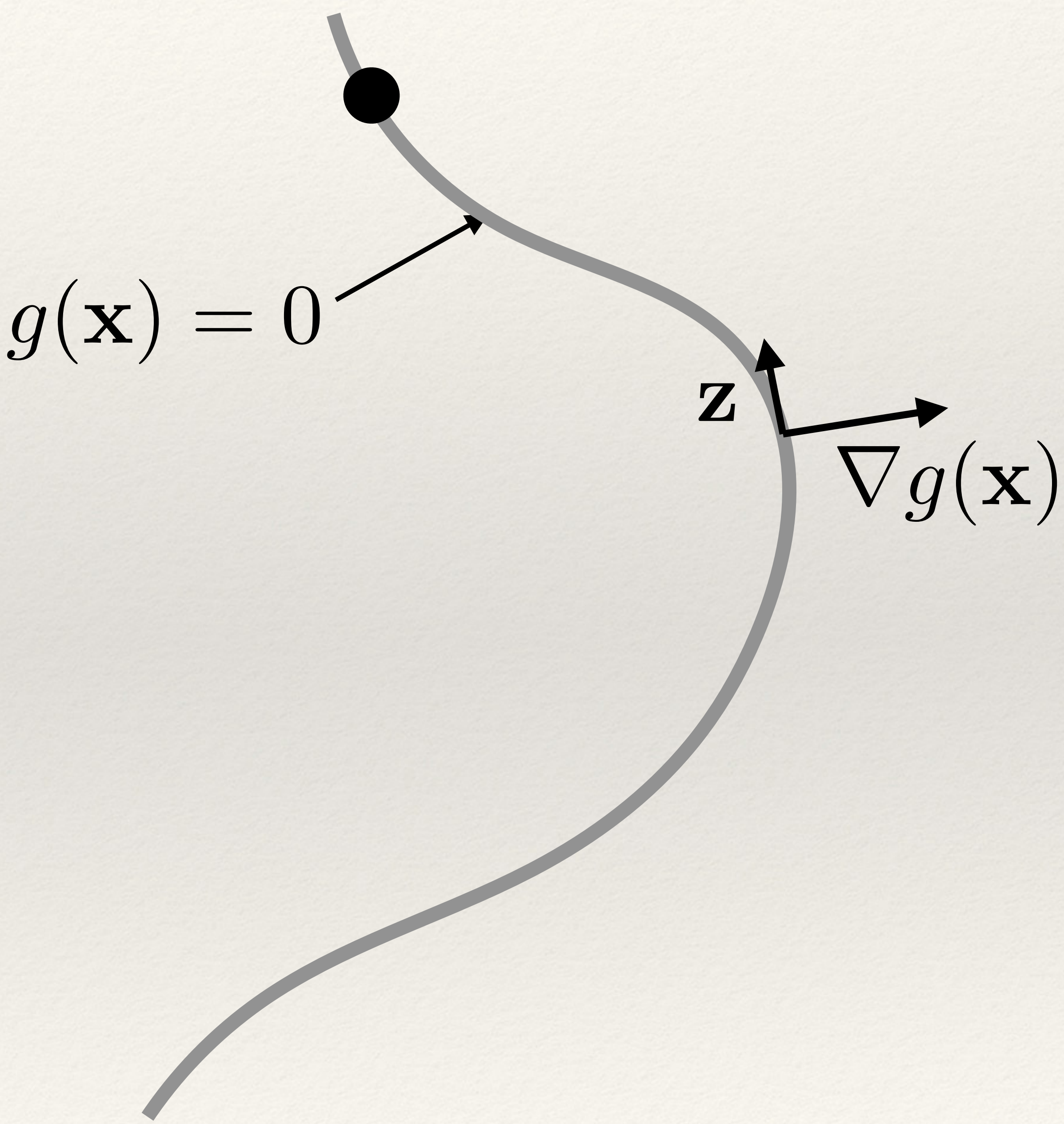
$$\mathbf{f}_c$$

- **Workless**

$$\mathbf{f}_c \cdot \mathbf{z} = 0$$

$$\Rightarrow \mathbf{f}_c = J^T \lambda$$

Lagrange multiplier



Constraint Force

- Constraint force

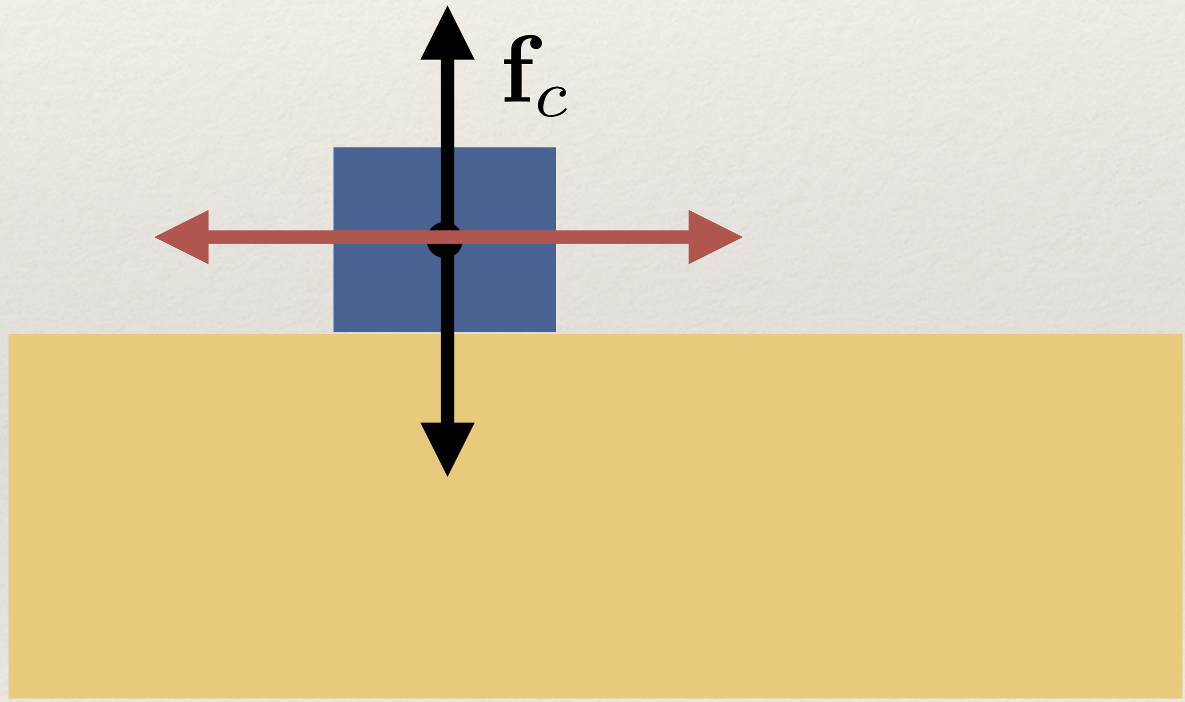
$$\mathbf{f}_c$$

- **Workless**

$$\mathbf{f}_c \cdot \mathbf{z} = 0$$

$$\Rightarrow \mathbf{f}_c = J^T \lambda$$

Lagrange multiplier



Equations of Motion

$$m\mathbf{a} = \mathbf{f} + J^T \lambda$$

$$g(\mathbf{x}) = 0$$

Differentiating the Constraint Equation

- Constraint tells us valid positions

$$\mathbf{g}(\mathbf{x}) = \mathbf{0}$$

- Differentiate to get valid velocities

$$\dot{\mathbf{g}}(\mathbf{x}) = J(\mathbf{x})\mathbf{v} = \mathbf{0}$$

- Differentiate again to get valid accelerations

$$\ddot{\mathbf{g}}(\mathbf{x}) = \dot{J}\mathbf{v} + J\mathbf{a} = \mathbf{0}$$

Equations of Motion

$$m\mathbf{a} = \mathbf{f} + J^T\lambda$$

$$g(\mathbf{x}) = 0$$

Equations of Motion

$$m\mathbf{a} = \mathbf{f} + J^T\lambda$$

$$\ddot{g}(\mathbf{x}) = \dot{J}\mathbf{v} + J\mathbf{a} = 0$$

Equations of Motion

$$m\mathbf{a} = \mathbf{f} + J^T \lambda$$

$$\ddot{g}(\mathbf{x}) = \dot{J}\mathbf{v} + J\mathbf{a} = 0$$

$$\Downarrow$$

KKT System

$$\begin{pmatrix} mI & -J^T \\ -J & 0 \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ -\dot{J}\mathbf{v} \end{pmatrix}$$

Multiple Constraints

- m constraints

$$\mathbf{g}(\mathbf{x}) = \begin{pmatrix} g_1(\mathbf{x}) \\ g_2(\mathbf{x}) \\ \vdots \\ g_m(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \mathbf{0}$$

- Constraint force

$$\mathbf{F}_c = J^T \lambda = \begin{pmatrix} \nabla g_1 & \nabla g_2 & \dots & \nabla g_m \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{pmatrix}$$

- Equations of motion

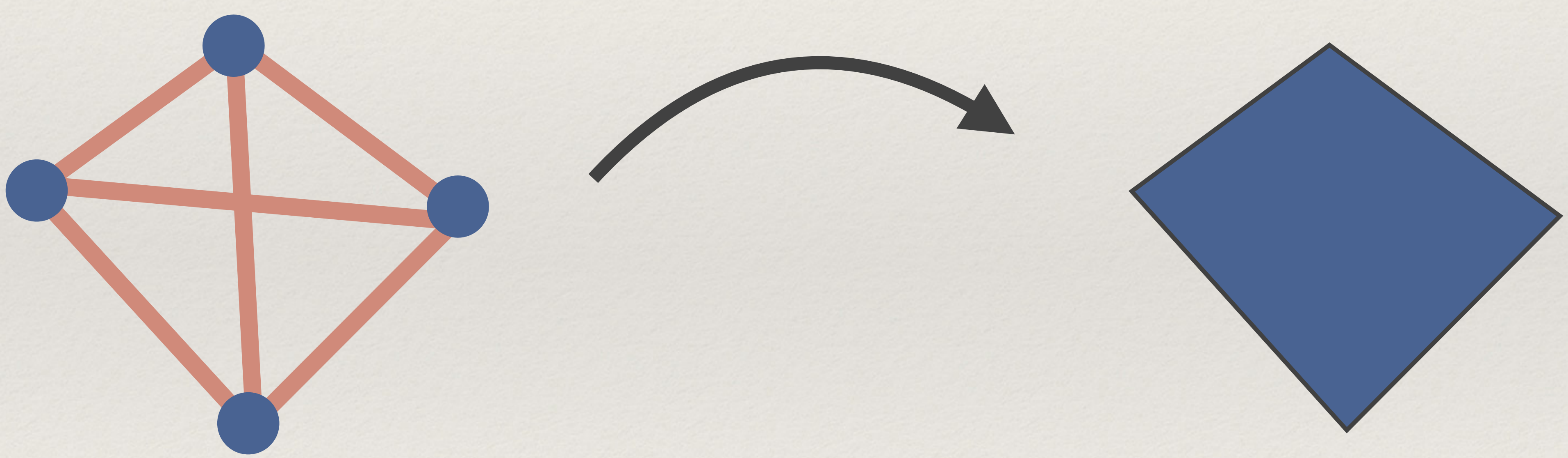
$$\begin{pmatrix} M & -J^T \\ -J & 0 \end{pmatrix} \begin{pmatrix} \mathbf{A} \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{F} \\ -\dot{J}\mathbf{V} \end{pmatrix}$$

Generalized Coordinates

- Instead of maximal coordinates $x_1, x_2, \ldots, x_n$ along with auxiliary conditions and forces
- Generalized coordinates $q_1, q_2, \ldots, q_N$ with $N < n$ that take constraints into account

Generalized Coordinates

- Example: rigid body



$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$

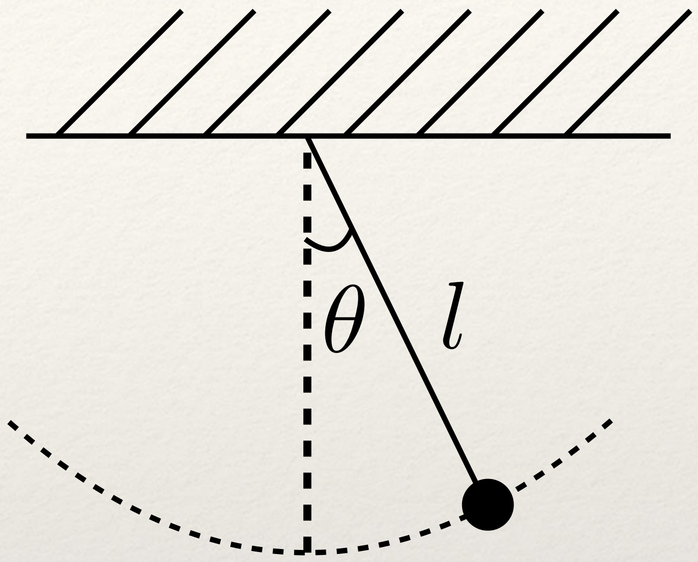
+ constraints

$\mathbf{x}, R$

Example: Pendulum

- Transformation equations

$$x(\theta) = l\sin\theta$$
$$y(\theta) = -l\cos\theta$$



- Velocity

$$\begin{pmatrix} \dot{x}(\theta) \\ \dot{y}(\theta) \end{pmatrix} = \begin{pmatrix} l\cos\theta\dot{\theta} \\ l\sin\theta\dot{\theta} \end{pmatrix} = J\dot{\theta}$$

- Lagrangian

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta$$

Equations of motion

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

Constrained Rigid Body Systems

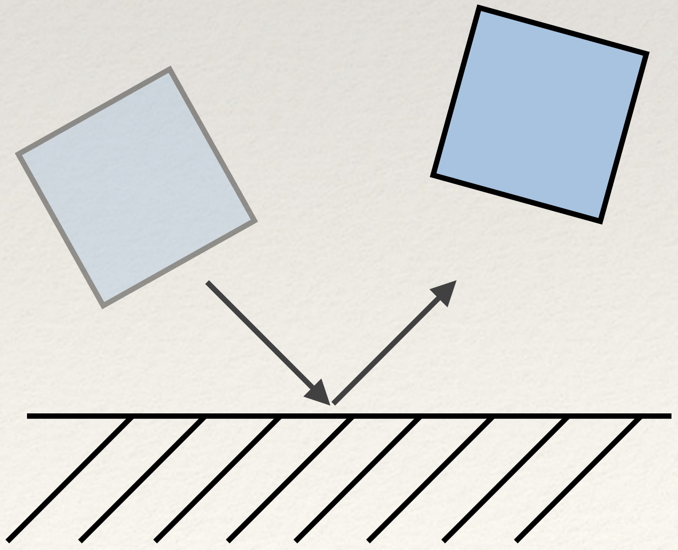
- Types of constraints
  - Articulation (joints)
  - Collisions
  - Resting and sliding contact

Impulse-Momentum Equations

- Acceleration-level formulation doesn't work well with discontinuous velocities, frictional contact
- Instead, integrate f = ma to get **impulse-momentum** formulation

$$\int_{t_1}^{t_2} m\mathbf{a}\,dt = \int_{t_1}^{t_2} \mathbf{f}\,dt,$$

$$\Rightarrow m(\mathbf{v}(t_2) - \mathbf{v}(t_1)) = \mathbf{j}$$



Impulse-Momentum Equations

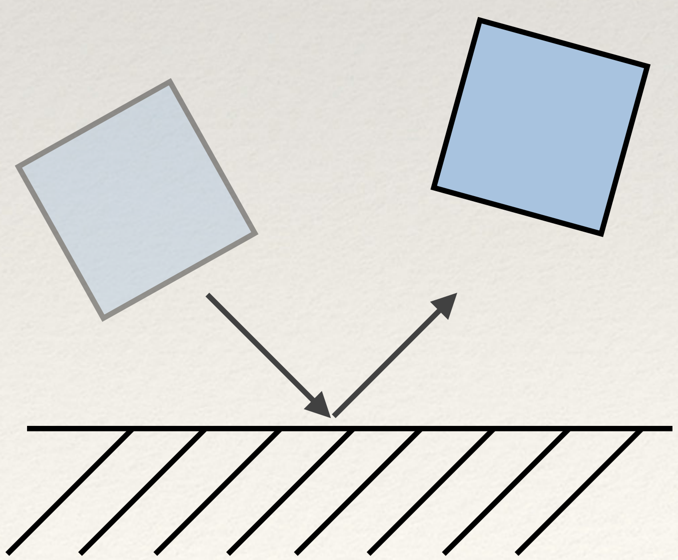
- Integrated, semi-discrete equations

$$M\mathbf{V}^{n+1} = M\mathbf{V}^{n} + \triangle t\mathbf{F} + J^{T}\mu^{n+1}$$

- Combine with velocity-level constraint equation J**V** = **0**

$$\begin{pmatrix} M & -J^T \\ -J & 0 \end{pmatrix} \begin{pmatrix} \mathbf{V}^{n+1} \\ \mu^{n+1} \end{pmatrix} = \begin{pmatrix} M\mathbf{V}^{n} + \Delta t\mathbf{F} \\ \mathbf{0} \end{pmatrix}$$

$$\Rightarrow JM^{-1}J^{T}\mu^{n+1} = -J\mathbf{V}^{n} - \Delta tJM^{-1}\mathbf{F}$$



Impulse-Momentum Equations

- Instead of solving global, coupled system, common to split

update with non-constraint forces

$$M\mathbf{V}^{\star} = M\mathbf{V}^{n} + \Delta t\mathbf{F}$$

add constraint impulses

$$M\mathbf{V}^{n+1} = M\mathbf{V}^{\star} + J^{T}\mu^{n+1}$$

Impulse-Momentum Equations

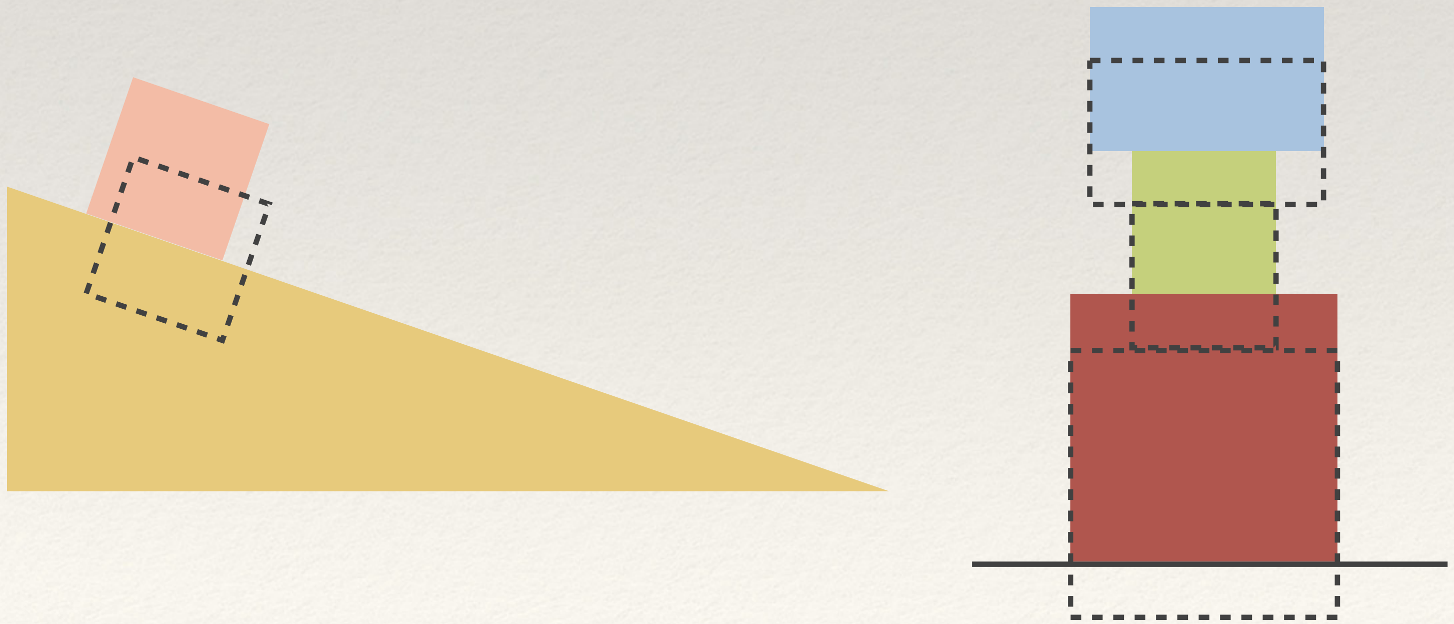
- Instead of solving global, coupled system, common to split

update with non-constraint forces

$$M\mathbf{V}^{\star} = M\mathbf{V}^{n} + \Delta t\mathbf{F}$$

add constraint impulses

$$M\mathbf{V}^{n+1} = M\mathbf{V}^{\star} + J^{T}\mu^{n+1}$$



Impulse-Momentum Equations

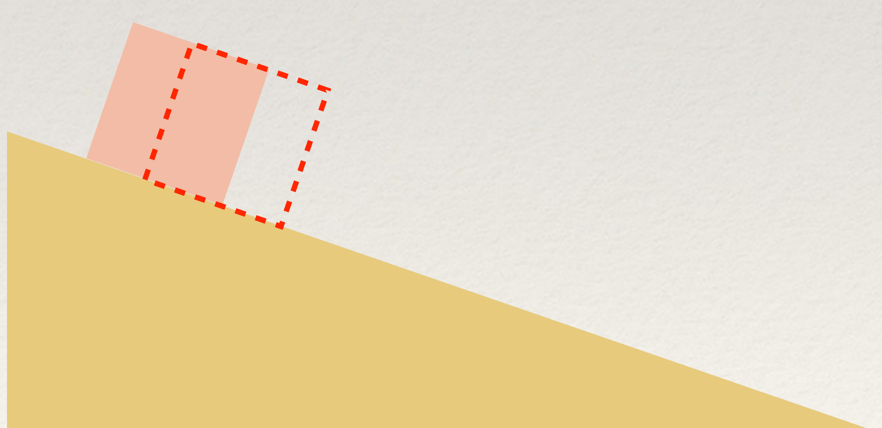
- Instead of solving global, coupled system, common to split

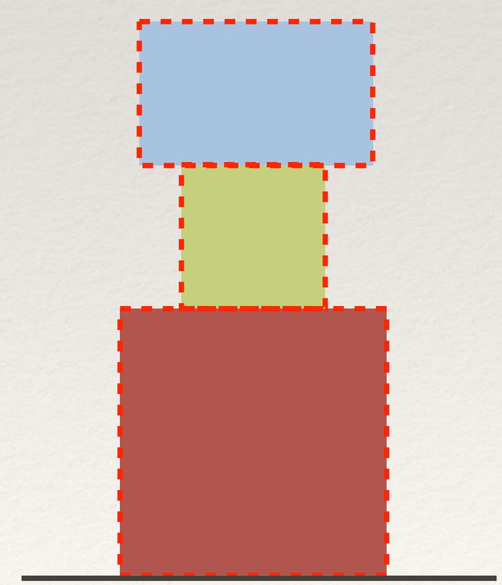
update with non-constraint forces

$$M\mathbf{V}^{\star} = M\mathbf{V}^{n} + \Delta t\mathbf{F}$$

add constraint impulses

$$M\mathbf{V}^{n+1} = M\mathbf{V}^{\star} + J^{T}\mu^{n+1}$$





global impulse solve

Impulse-Momentum Equations

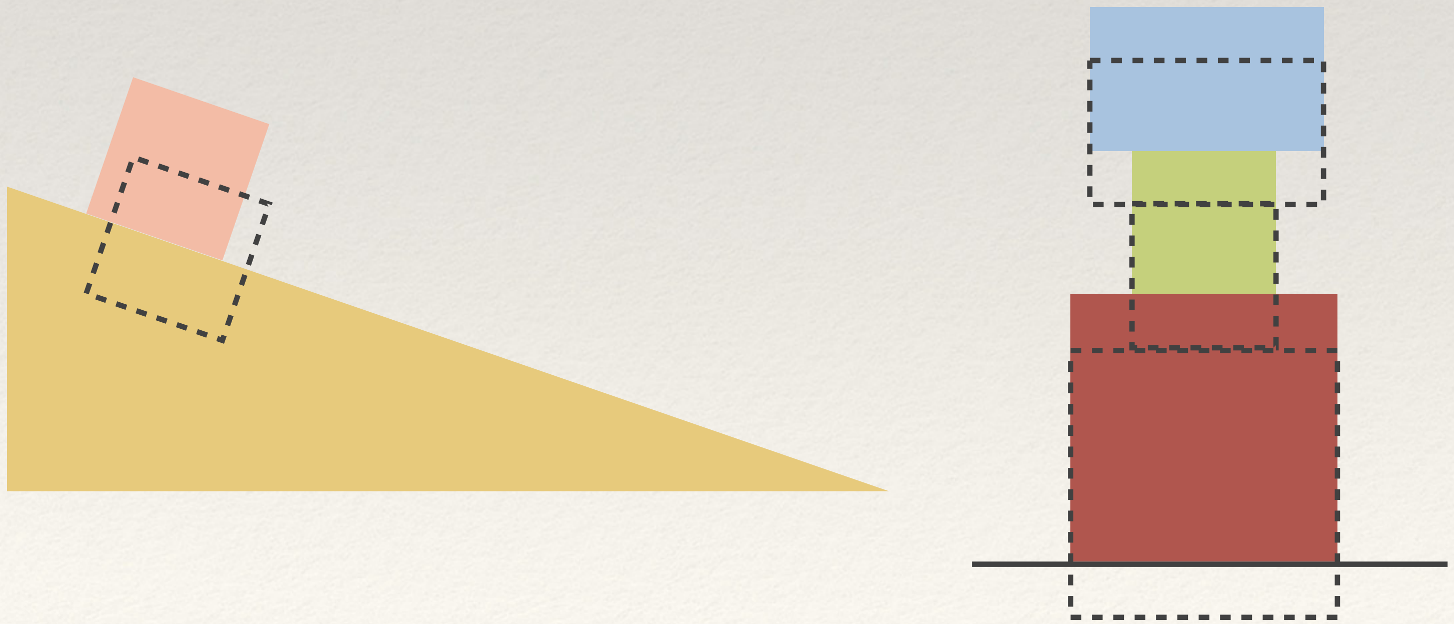
- Instead of solving global, coupled system, common to split

update with non-constraint forces

$$M\mathbf{V}^{\star} = M\mathbf{V}^{n} + \Delta t\mathbf{F}$$

add constraint impulses

$$M\mathbf{V}^{n+1} = M\mathbf{V}^{\star} + J^{T}\mu^{n+1}$$



iterative impulse solve

Impulse-Momentum Equations

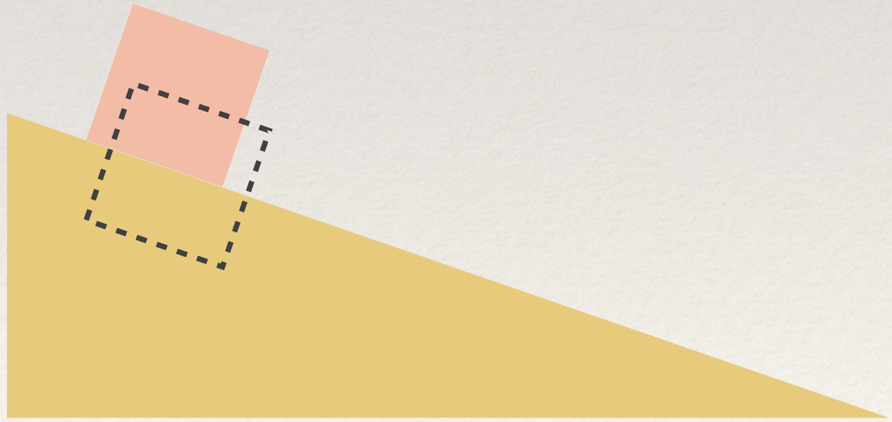
- Instead of solving global, coupled system, common to split

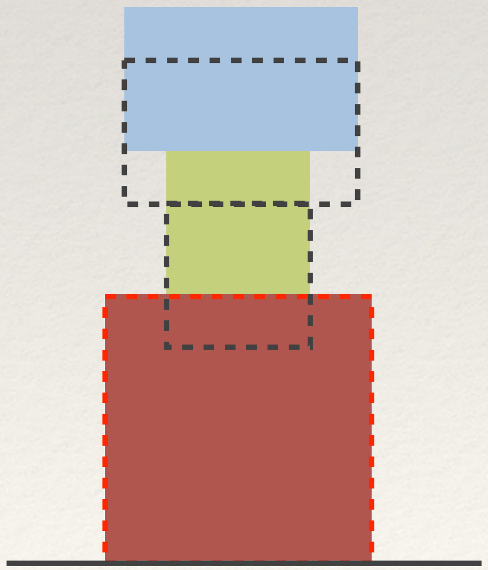
update with non-constraint forces

$$M\mathbf{V}^{\star} = M\mathbf{V}^{n} + \Delta t\mathbf{F}$$

add constraint impulses

$$M\mathbf{V}^{n+1} = M\mathbf{V}^{\star} + J^{T}\mu^{n+1}$$





iterative impulse solve

Impulse-Momentum Equations

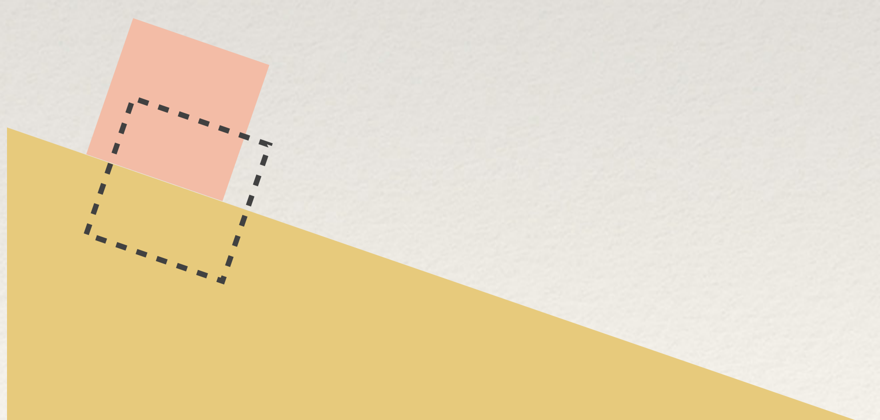
- Instead of solving global, coupled system, common to split

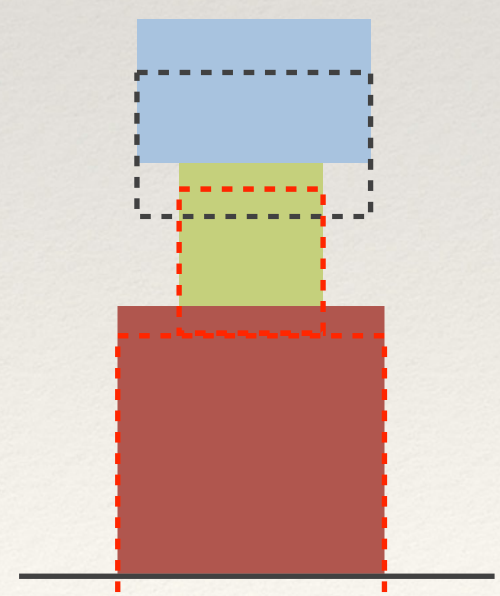
update with non-constraint forces

$$M\mathbf{V}^{\star} = M\mathbf{V}^{n} + \Delta t\mathbf{F}$$

add constraint impulses

$$M\mathbf{V}^{n+1} = M\mathbf{V}^{\star} + J^{T}\mu^{n+1}$$





iterative impulse solve

Impulse-Momentum Equations

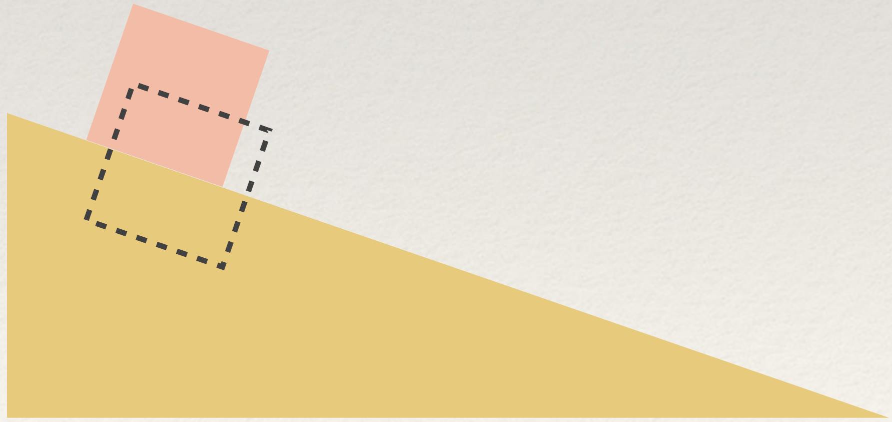
- Instead of solving global, coupled system, common to split

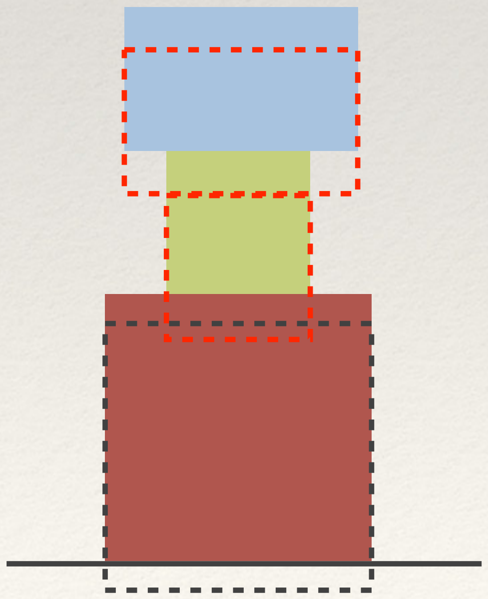
update with non-constraint forces

$$M\mathbf{V}^{\star} = M\mathbf{V}^{n} + \Delta t \mathbf{F}$$

add constraint impulses

$$M\mathbf{V}^{n+1} = M\mathbf{V}^{\star} + J^{T}\mu^{n+1}$$





iterative impulse solve

Impulse-Momentum Equations

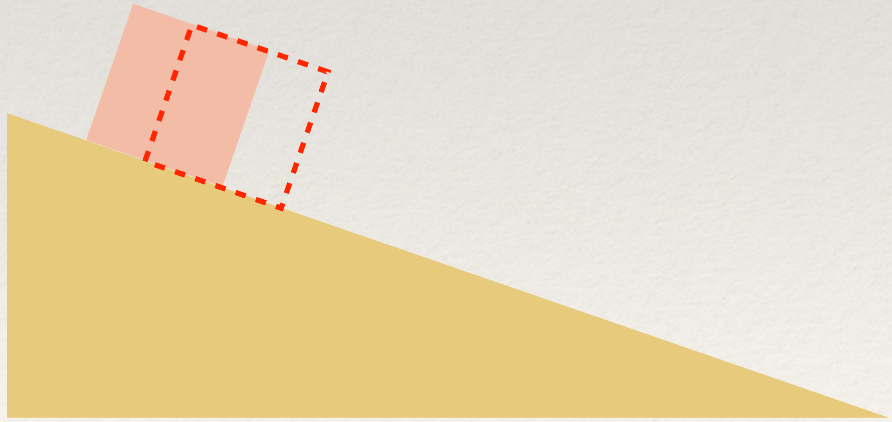
- Instead of solving global, coupled system, common to split

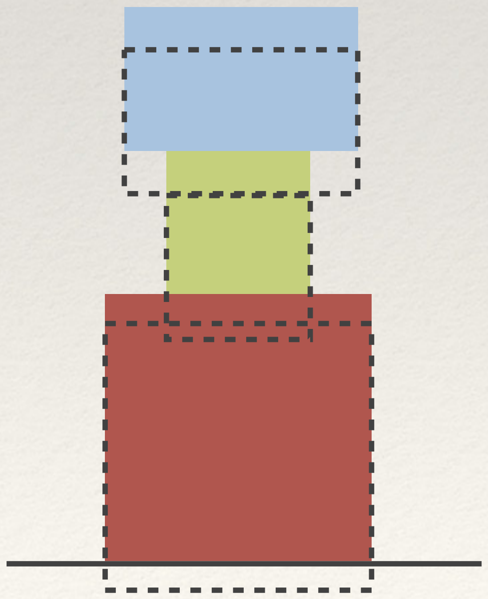
update with non-constraint forces

$$M\mathbf{V}^{\star} = M\mathbf{V}^{n} + \Delta t\mathbf{F}$$

add constraint impulses

$$M\mathbf{V}^{n+1} = M\mathbf{V}^{\star} + J^{T}\mu^{n+1}$$





iterative impulse solve

Impulse-Momentum Equations

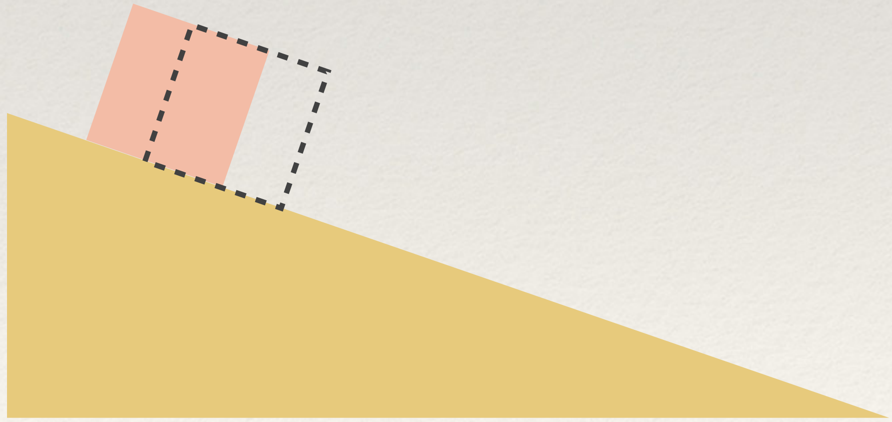
- Instead of solving global, coupled system, common to split

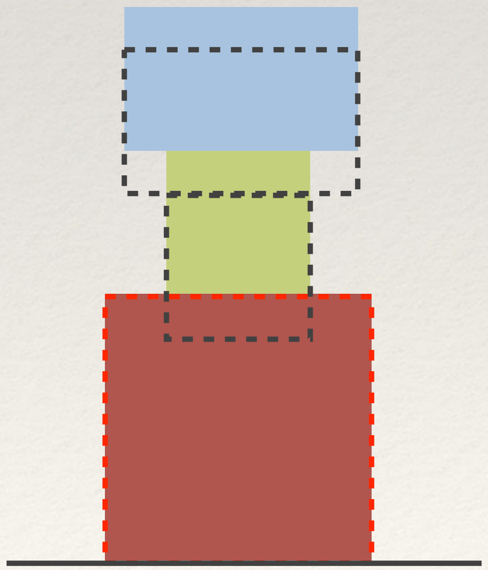
update with non-constraint forces

$$M\mathbf{V}^{\star} = M\mathbf{V}^{n} + \Delta t\mathbf{F}$$

add constraint impulses

$$M\mathbf{V}^{n+1} = M\mathbf{V}^{\star} + J^{T}\mu^{n+1}$$





iterative impulse solve

Impulse-Momentum Equations

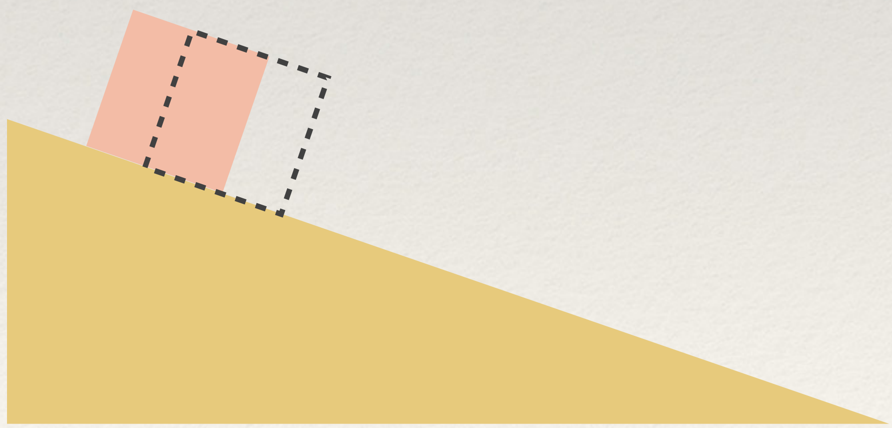
- Instead of solving global, coupled system, common to split

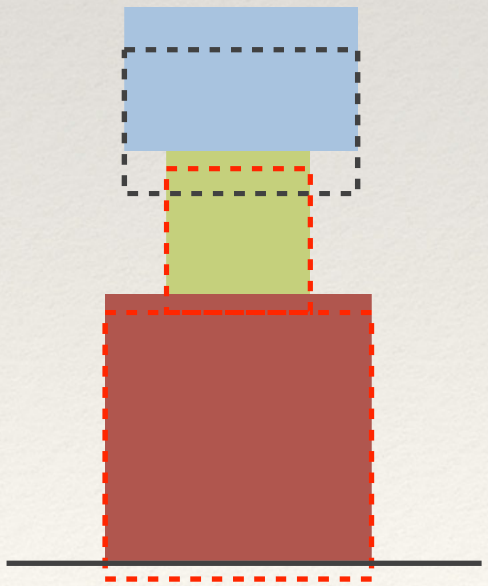
update with non-constraint forces

$$M\mathbf{V}^{\star} = M\mathbf{V}^{n} + \Delta t\mathbf{F}$$

add constraint impulses

$$M\mathbf{V}^{n+1} = M\mathbf{V}^{\star} + J^{T}\mu^{n+1}$$





iterative impulse solve

Impulse-Momentum Equations

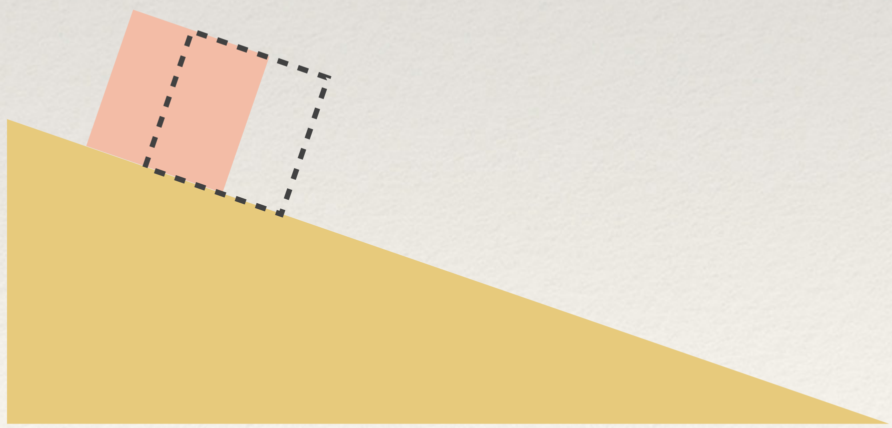
- Instead of solving global, coupled system, common to split

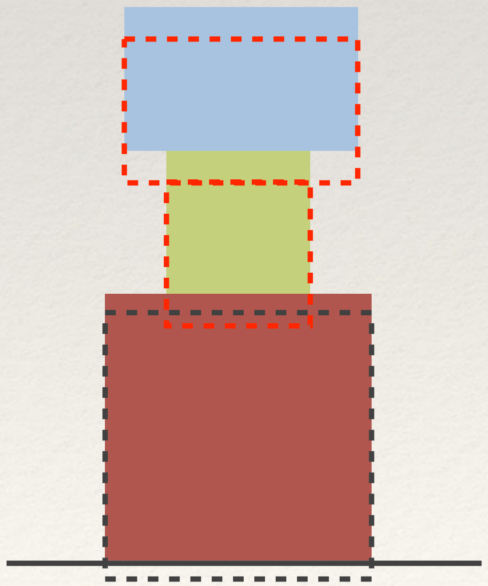
update with non-constraint forces

$$M\mathbf{V}^{\star} = M\mathbf{V}^{n} + \Delta t\mathbf{F}$$

add constraint impulses

$$M\mathbf{V}^{n+1} = M\mathbf{V}^{\star} + J^{T}\mu^{n+1}$$





iterative impulse solve

Impulse-Momentum Equations

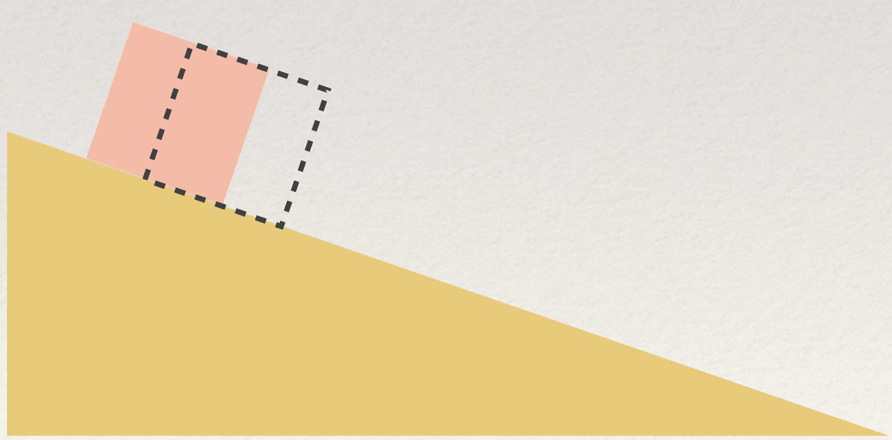
- Instead of solving global, coupled system, common to split

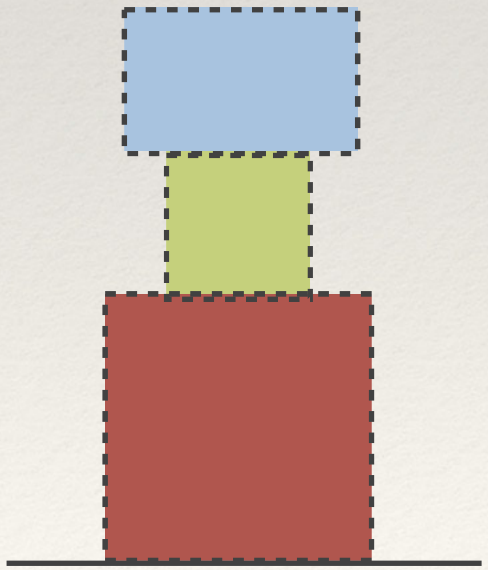
update with non-constraint forces

$$M\mathbf{V}^{\star} = M\mathbf{V}^{n} + \Delta t\mathbf{F}$$

add constraint impulses

$$M\mathbf{V}^{n+1} = M\mathbf{V}^{\star} + J^{T}\mu^{n+1}$$



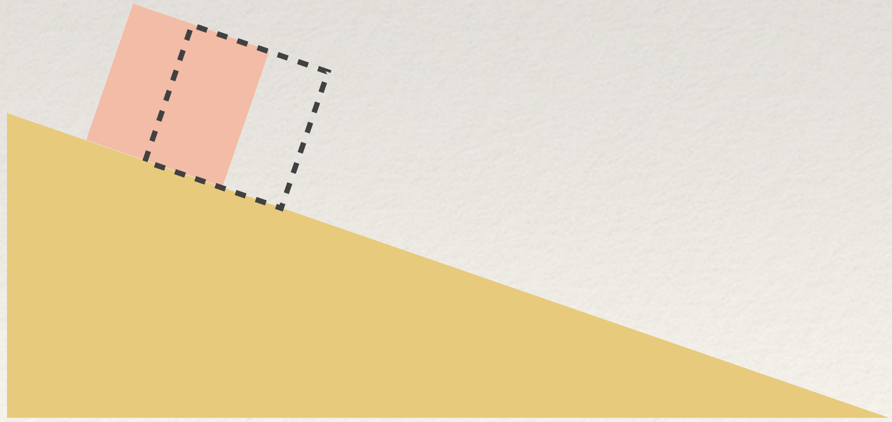


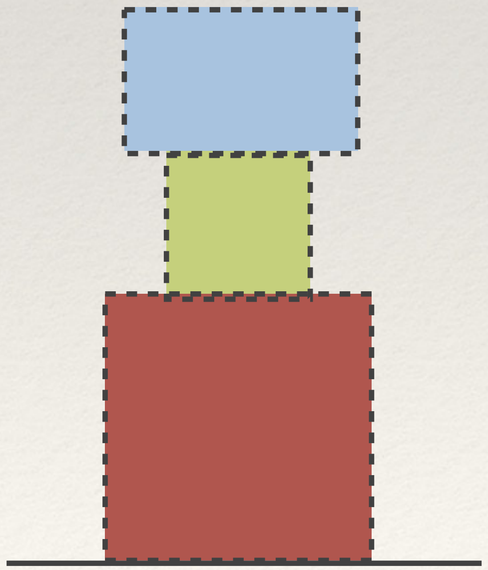
iterative impulse solve

repeat fixed number of times or until tolerance met

Global vs. Iterative Solve

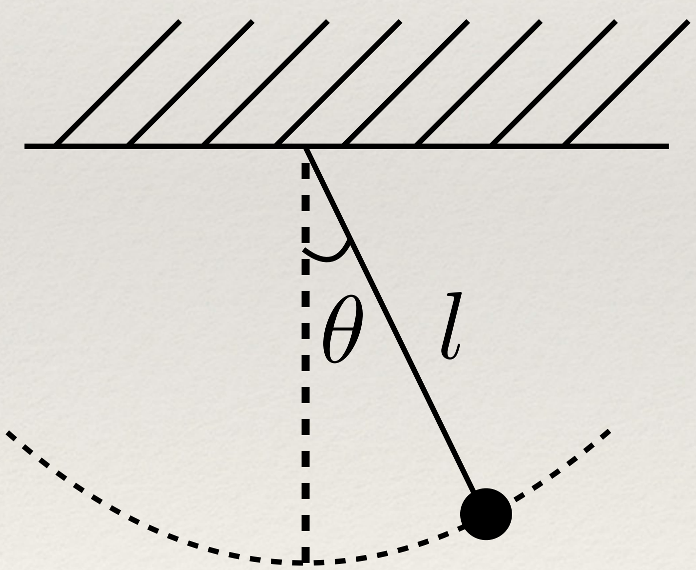
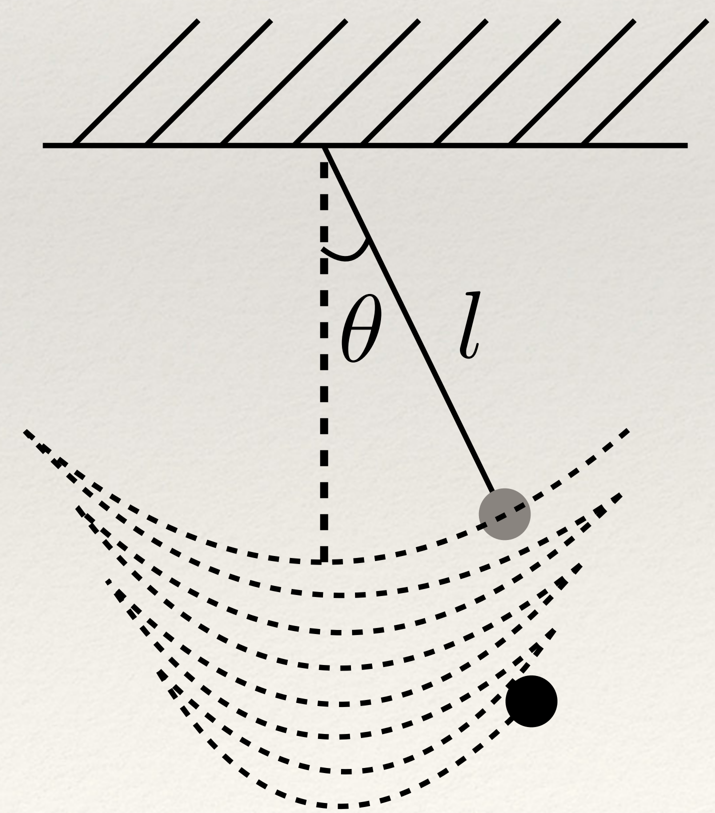
- Global
  - need to solve larger linear system
  - LCP for inequality constraints
- Iterative
  - may be slow to converge
  - simple to do inequality constraints





Handling Drift With Stabilization

- Approach was based on velocity-level constraints lead to drift in positions $\begin{pmatrix} M & -J^T \\ -J & 0 \end{pmatrix} \begin{pmatrix} \mathbf{V}^{n+1} \\ \mu^{n+1} \end{pmatrix} = \begin{pmatrix} M\mathbf{V}^n + \Delta t\mathbf{F} \\ \mathbf{0} \end{pmatrix}$

Exact	**With drift**
	

Handling Drift With Stabilization

- Approach was based on velocity-level constraints lead to drift in positions $\begin{pmatrix} M & -J^T \\ -J & 0 \end{pmatrix} \begin{pmatrix} \mathbf{V}^{n+1} \\ \mu^{n+1} \end{pmatrix} = \begin{pmatrix} M\mathbf{V}^n + \Delta t\mathbf{F} \\ \mathbf{0} \end{pmatrix}$
- Correct drift with **stabilization**
  - E.g., Baumgarte stabilization

$$\dot{\mathbf{g}}(\mathbf{x}) + \gamma \mathbf{g}(\mathbf{x}) = 0$$

  - Post-stabilization - modify positions so they satisfy constraints after time step

Softening Constraints

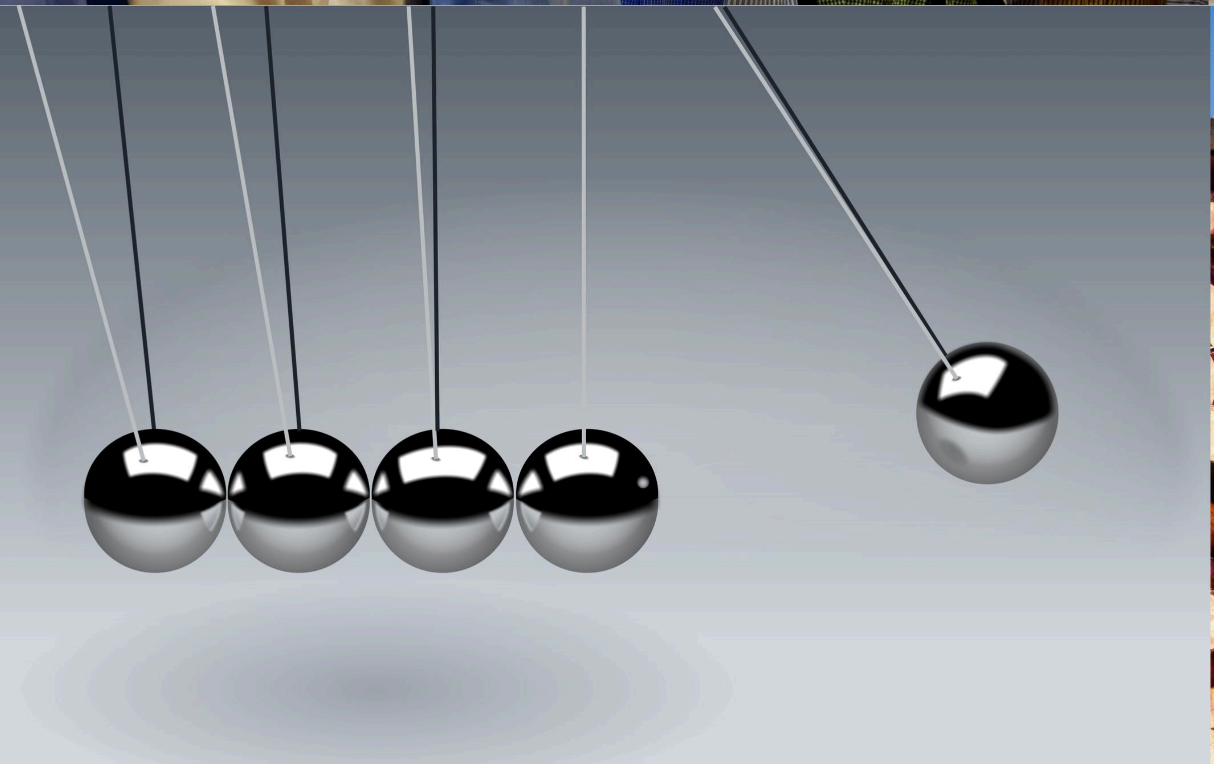
- Also common to soften constraints

$$\begin{pmatrix} M & -J^T \\ -J & \gamma I \end{pmatrix} \begin{pmatrix} \mathbf{V}^{n+1} \\ \mu^{n+1} \end{pmatrix} = \begin{pmatrix} M\mathbf{V}^n + \Delta t \mathbf{F} \\ -\frac{\beta}{\Delta t}\mathbf{g}(\mathbf{X}^n) \end{pmatrix}$$

- Stabilizes constraints
- Regularizes system
  - Better numerical properties
  - Handle redundant constraints
- Adds some compliance to constraint

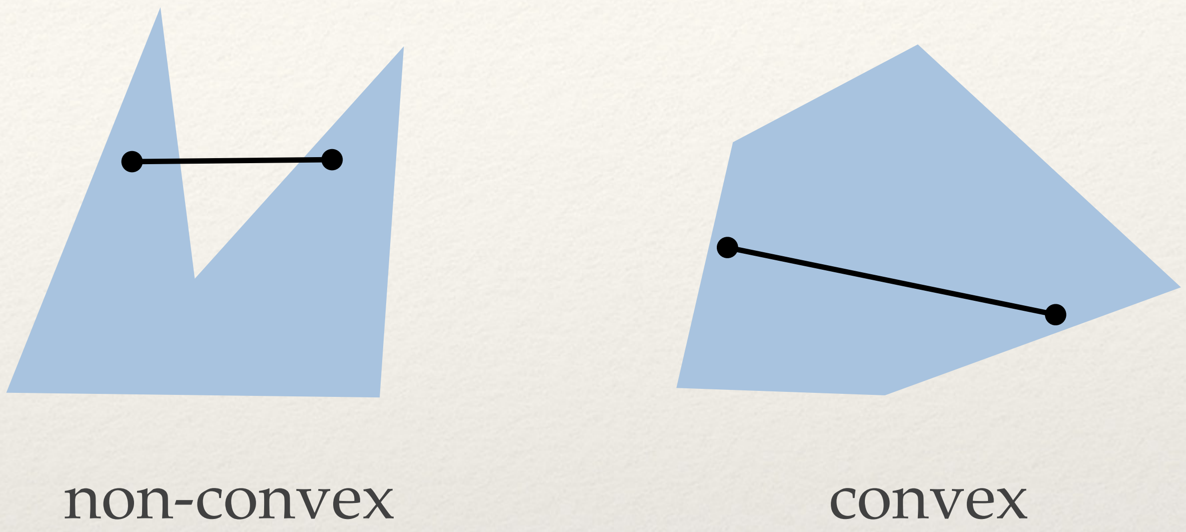
Collisions and Contact



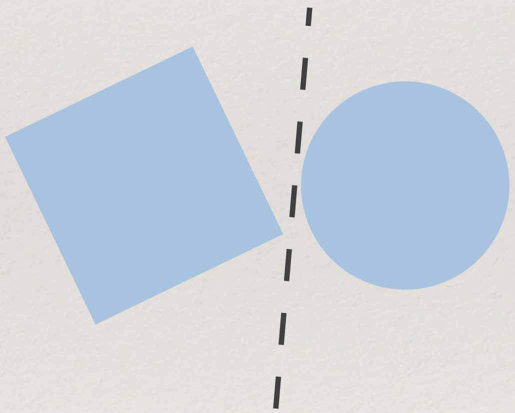



Collision Detection

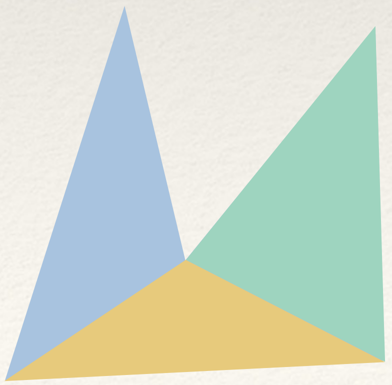
- Polygonal geometry



- Separating axis theorem



- Convex decomposition



Collision Detection

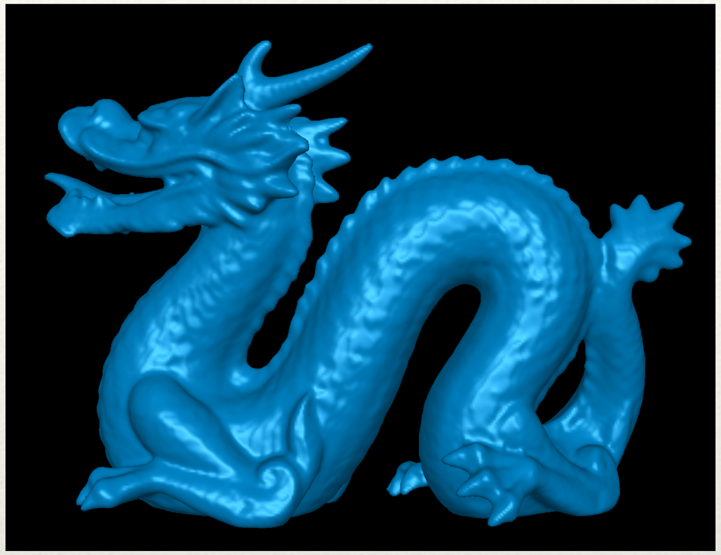
- Signed distance field

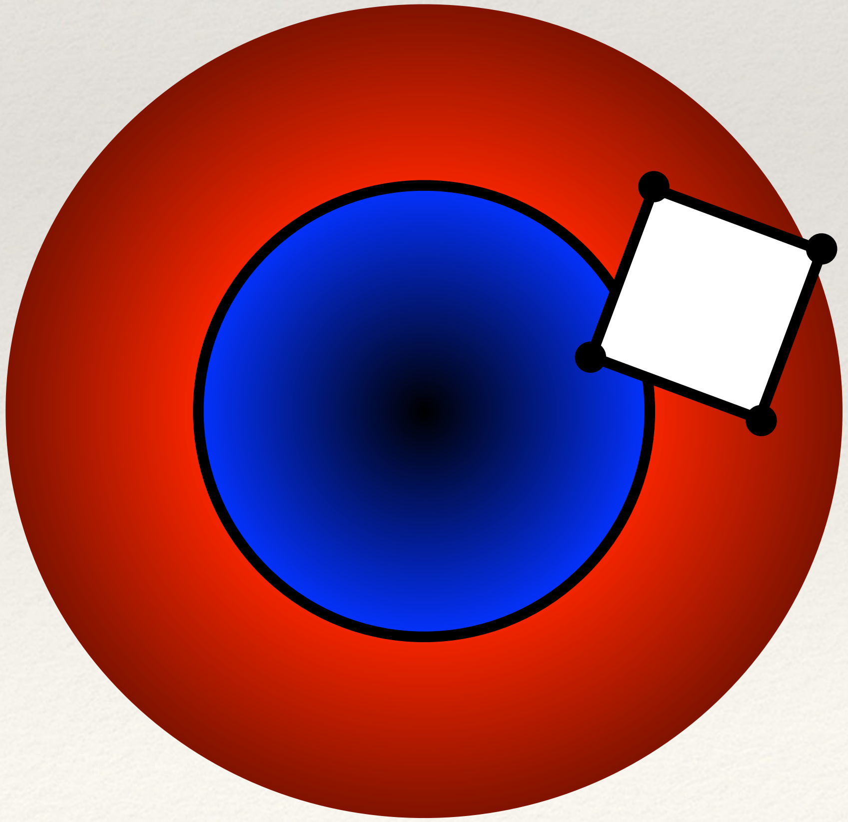
$$\phi(\mathbf{x})$$

- Zero level set

$$\phi(\mathbf{x}) = 0$$

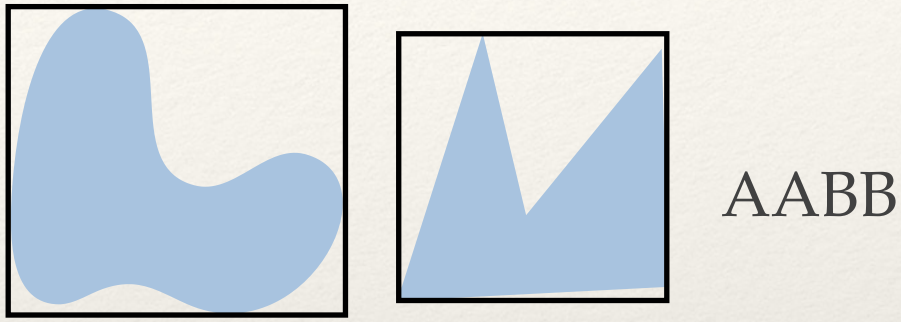
- Fast inside/outside tests
- Penetration depth $\phi(\mathbf{x})$
- Normals $\nabla\phi(\mathbf{x})$



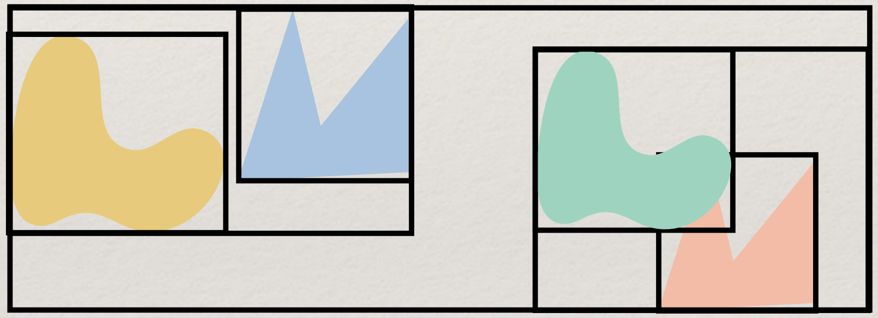


Accelerating Collision Detection

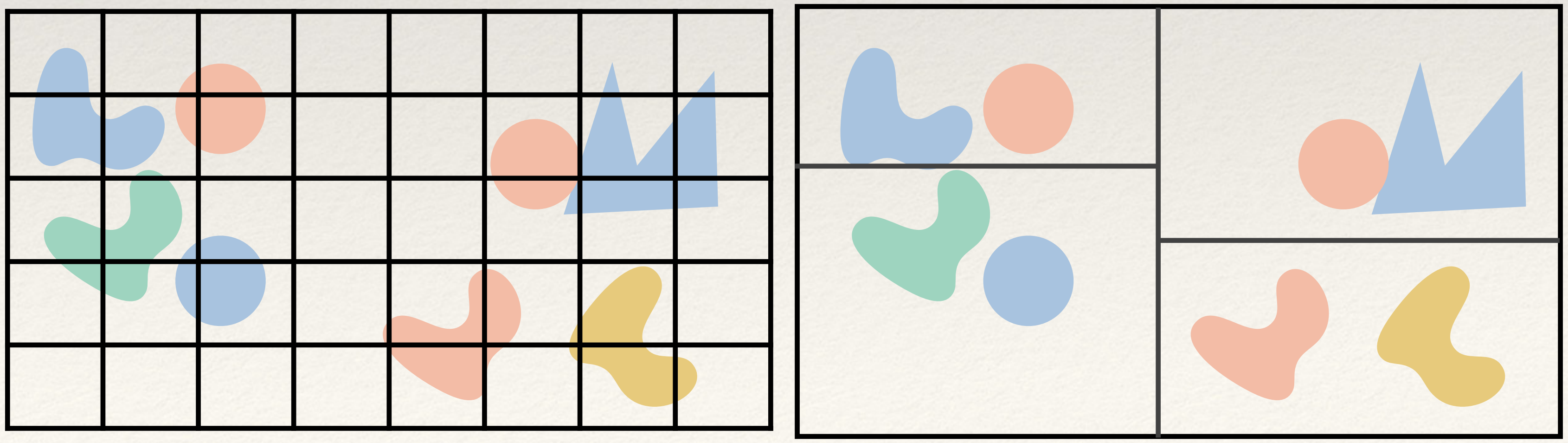
- Bounding volumes



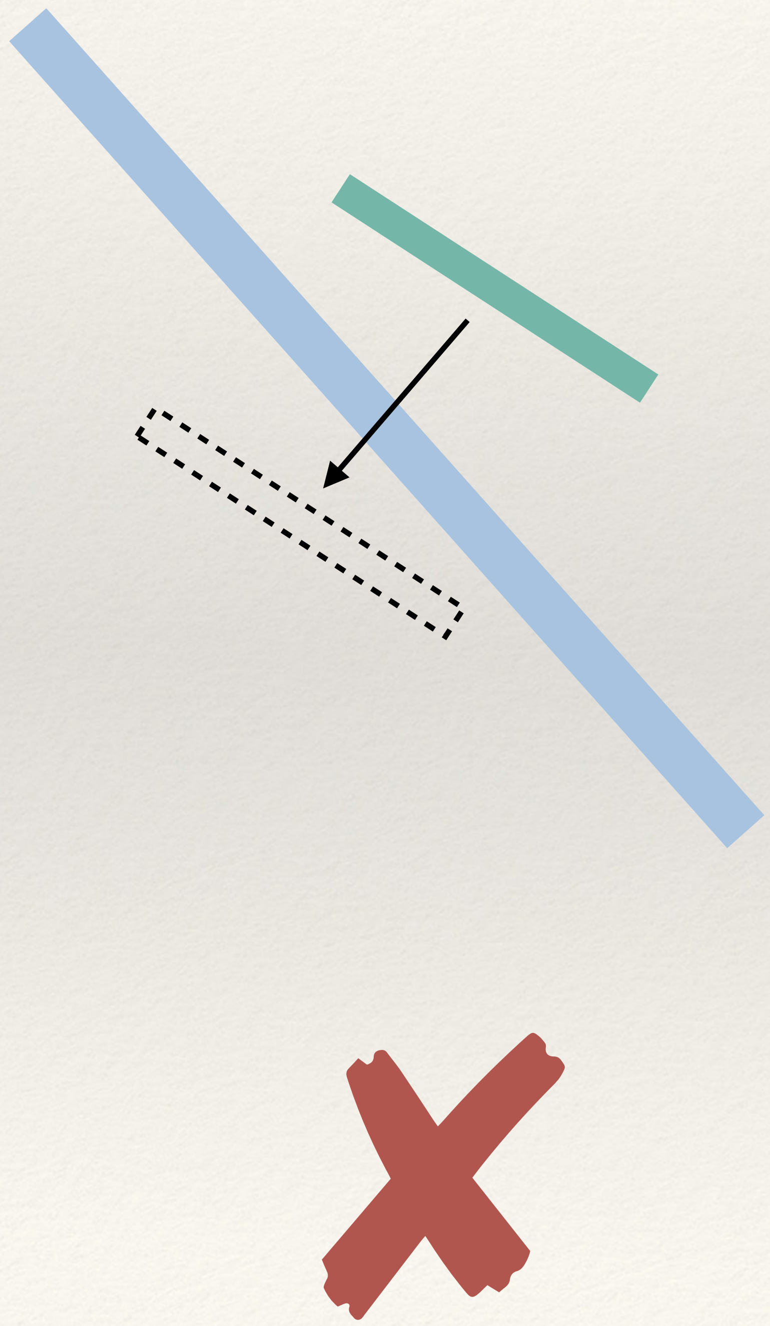
- Hierarchical bounding volumes

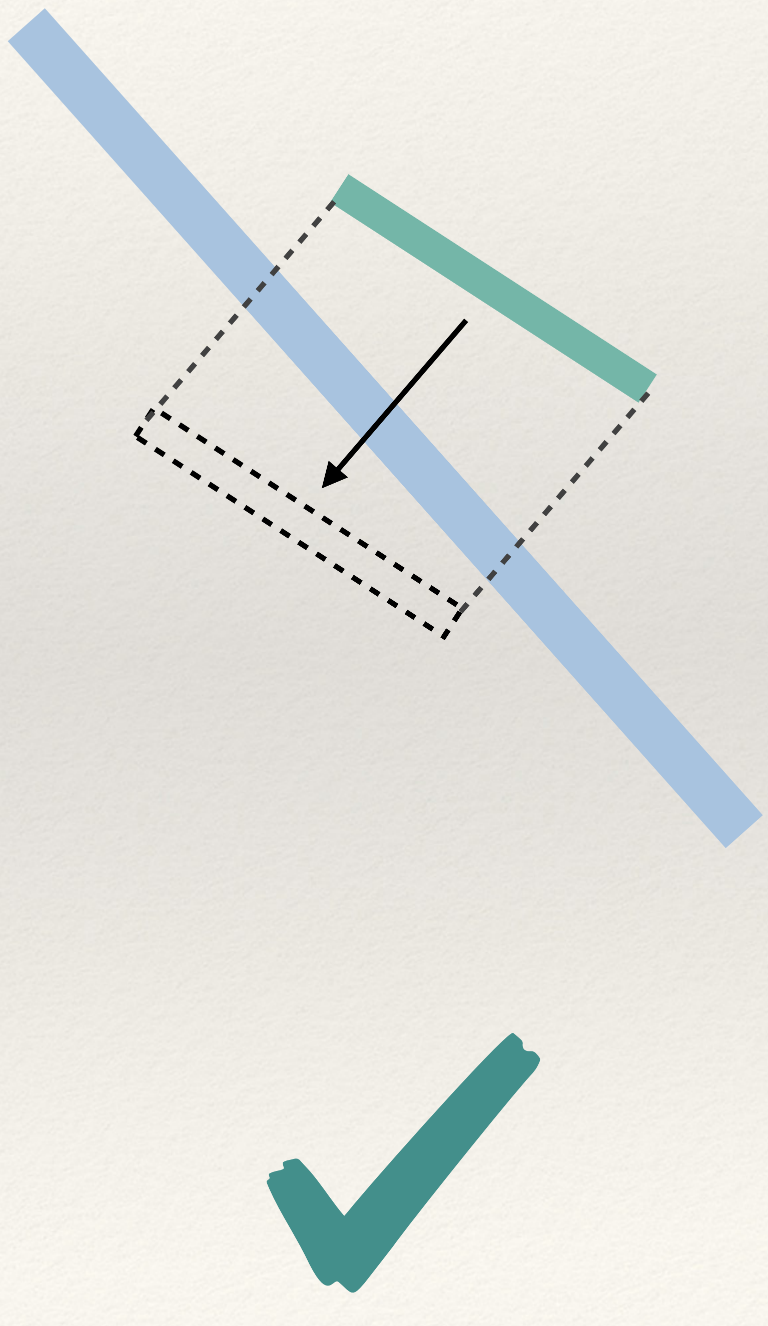


- Spatial partitions



Discrete vs. Continuous Collision Detection

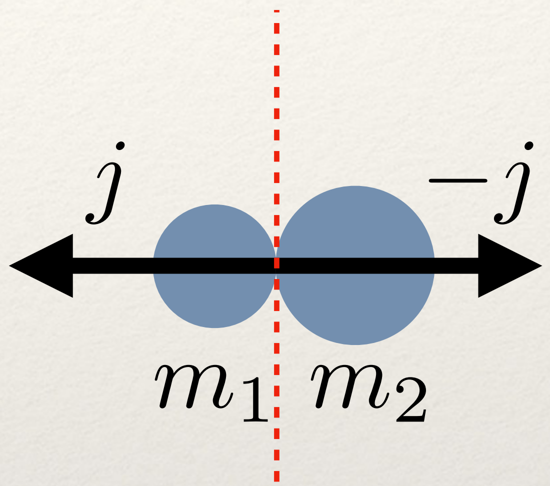




Collision Response: Inelastic

- Newton's third law (action/reaction)

$$m_1 v_1' = m_1 v_1 + j$$
$$m_2 v_2' = m_2 v_2 - j$$



- Assume inelastic (sticking)

$$v_1' = v_2'$$

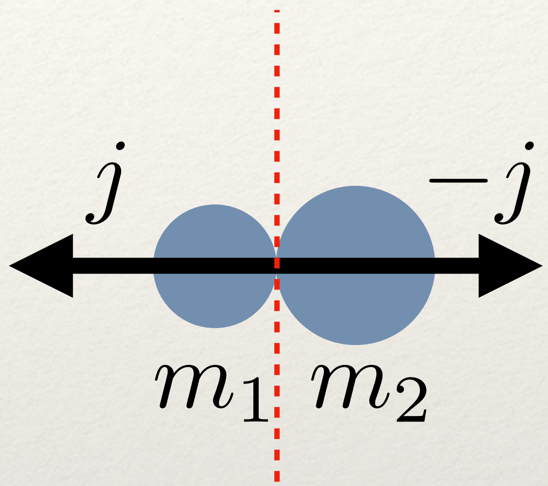
- Solve: $j = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)^{-1} (v_2 - v_1)$

$$v_1' = v_2' = \left(\frac{m_1}{m_1 + m_2}\right) v_1 + \left(\frac{m_2}{m_1 + m_2}\right) v_2$$

Collision Response: Elastic

- Newton's third law (action/reaction)

$$m_1 v_1' = m_1 v_1 + j$$
$$m_2 v_2' = m_2 v_2 - j$$



- Assume elastic (bouncing)

$$(v_2' - v_1') = -\epsilon(v_2 - v_1)$$

coefficient of restitution

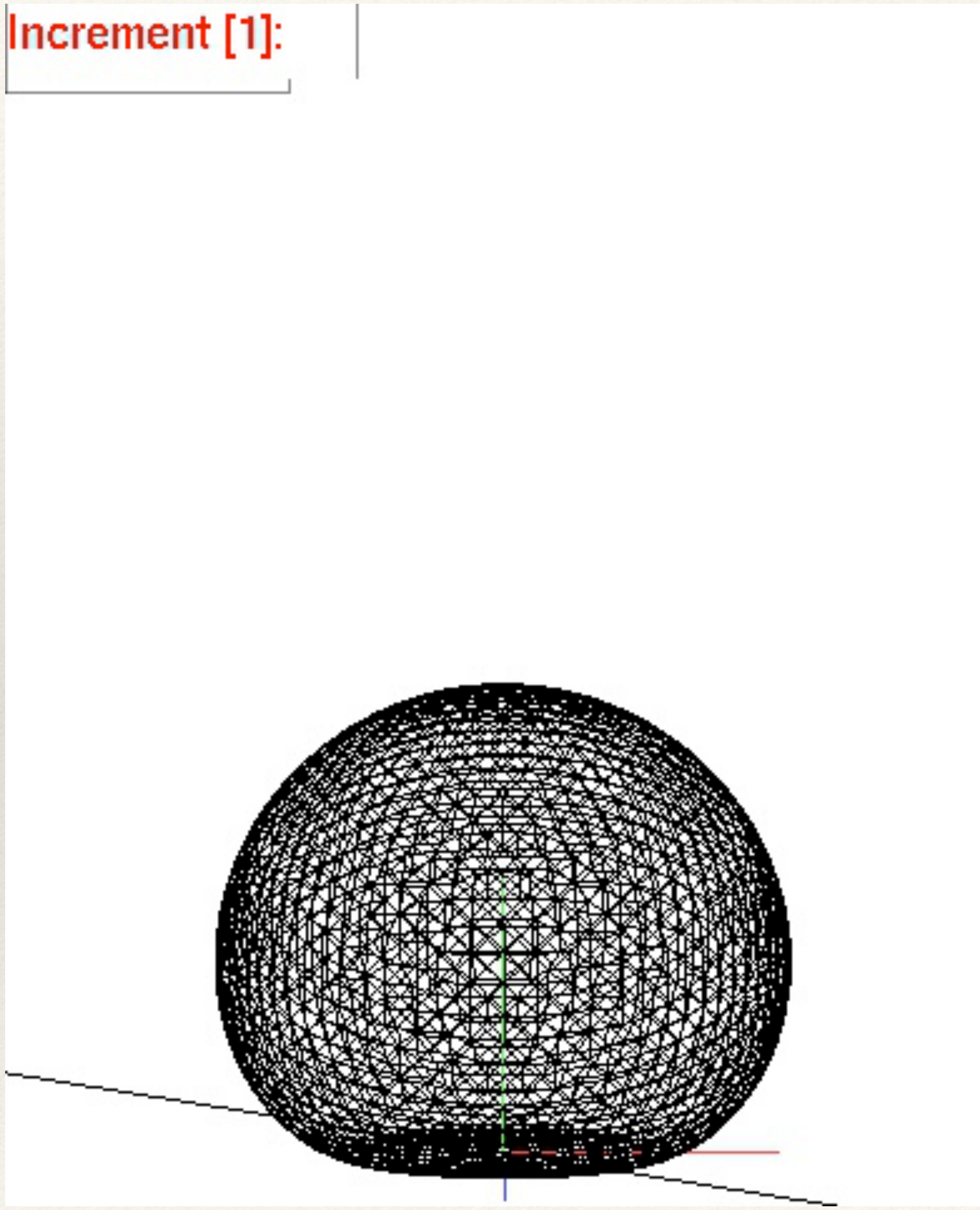
- Solve: $j = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)^{-1} (1 + \epsilon)(v_2 - v_1)$

$$v_1' = v_1 + \frac{1}{m_1} j, \quad v_2' = v_2 - \frac{1}{m_2} j$$

Deformable Object Collisions

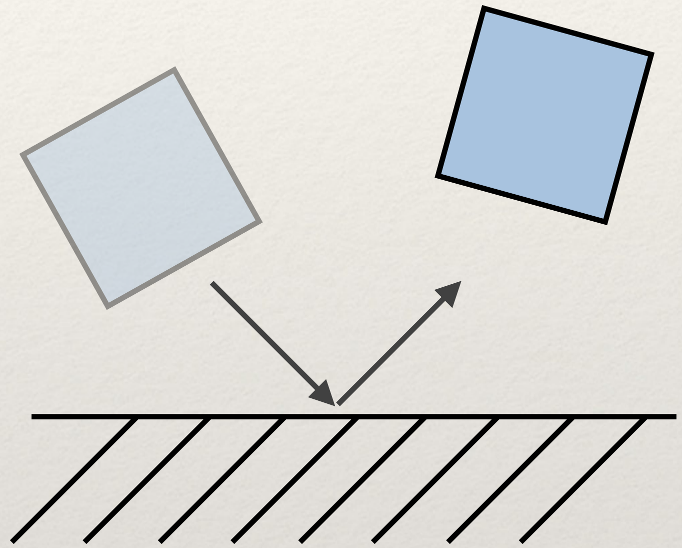
- Body compresses, stores energy and bounces
- Inelastic collision between particles and ground

$$m_1\mathbf{v}_1' = m_1\mathbf{v}_1 + \mathbf{j}$$
$$m_2\mathbf{v}_2' = m_2\mathbf{v}_2 - \mathbf{j}$$
$$\mathbf{v}_2' = \mathbf{v}_1'$$



Rigid Body Collisions

- Physically, similar to deformable body collisions
- But rigid idealization precludes storing energy
- Instead, algebraic collision laws for before / after collision
- Cases:
  - Inelastic (sticking)
  - Elastic: frictionless, with friction



Rigid Body Inelastic Collision

- Linear momentum

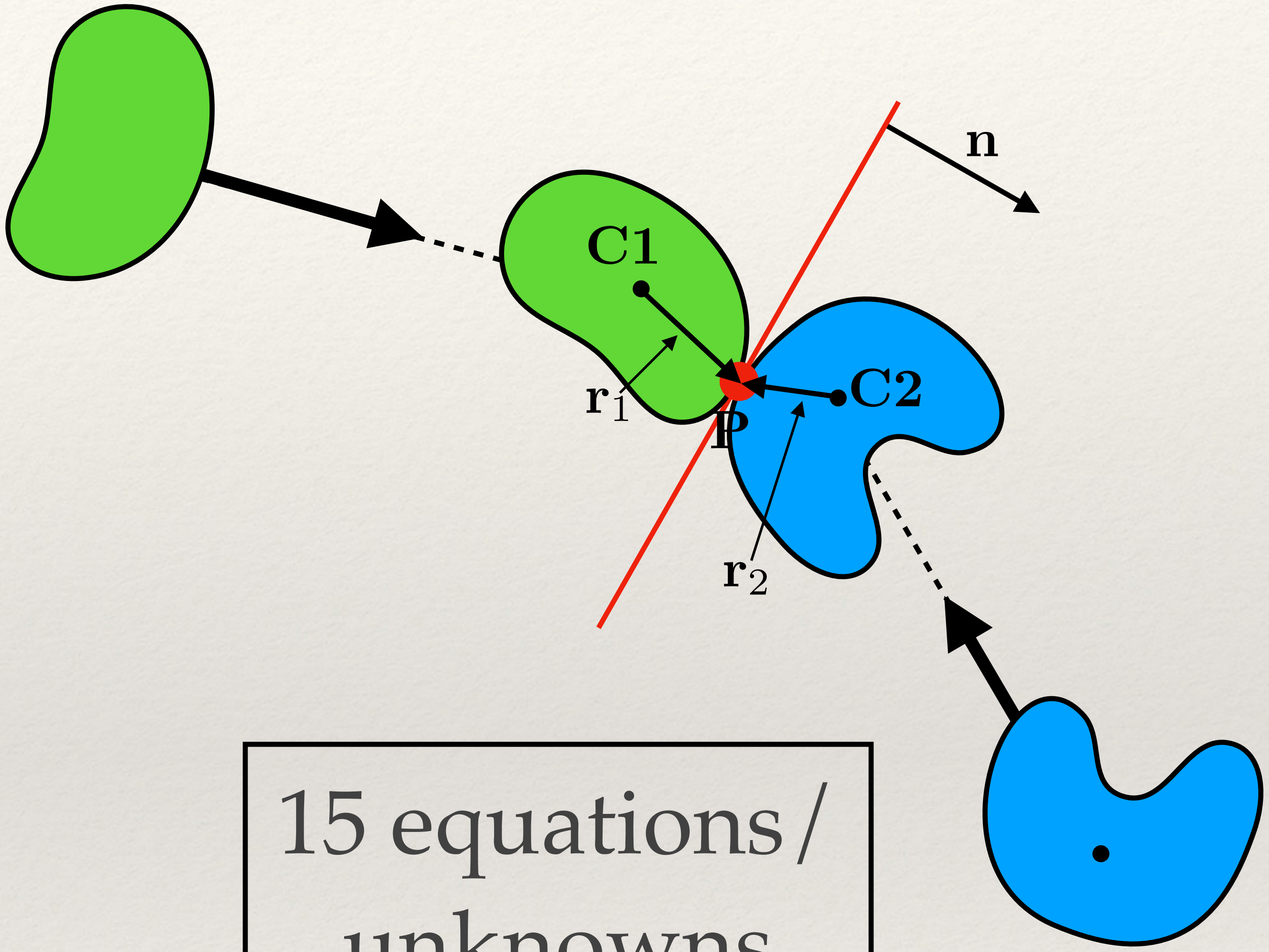
$$m_1\mathbf{v}'_{C1} = m_1\mathbf{v}_{C1} + \mathbf{j}$$
$$m_2\mathbf{v}'_{C2} = m_2\mathbf{v}_{C2} - \mathbf{j}$$

- Angular momentum

$$I_1\omega'_1 = I_1\omega_1 + \mathbf{r}_1 \times \mathbf{j}$$
$$I_2\omega'_2 = I_2\omega_2 - \mathbf{r}_2 \times \mathbf{j}$$

- Sticking

$$\mathbf{v}'_{P1} = \mathbf{v}'_{P2}$$



15 equations / unknowns

Rigid Body Frictionless Collision

- Linear momentum

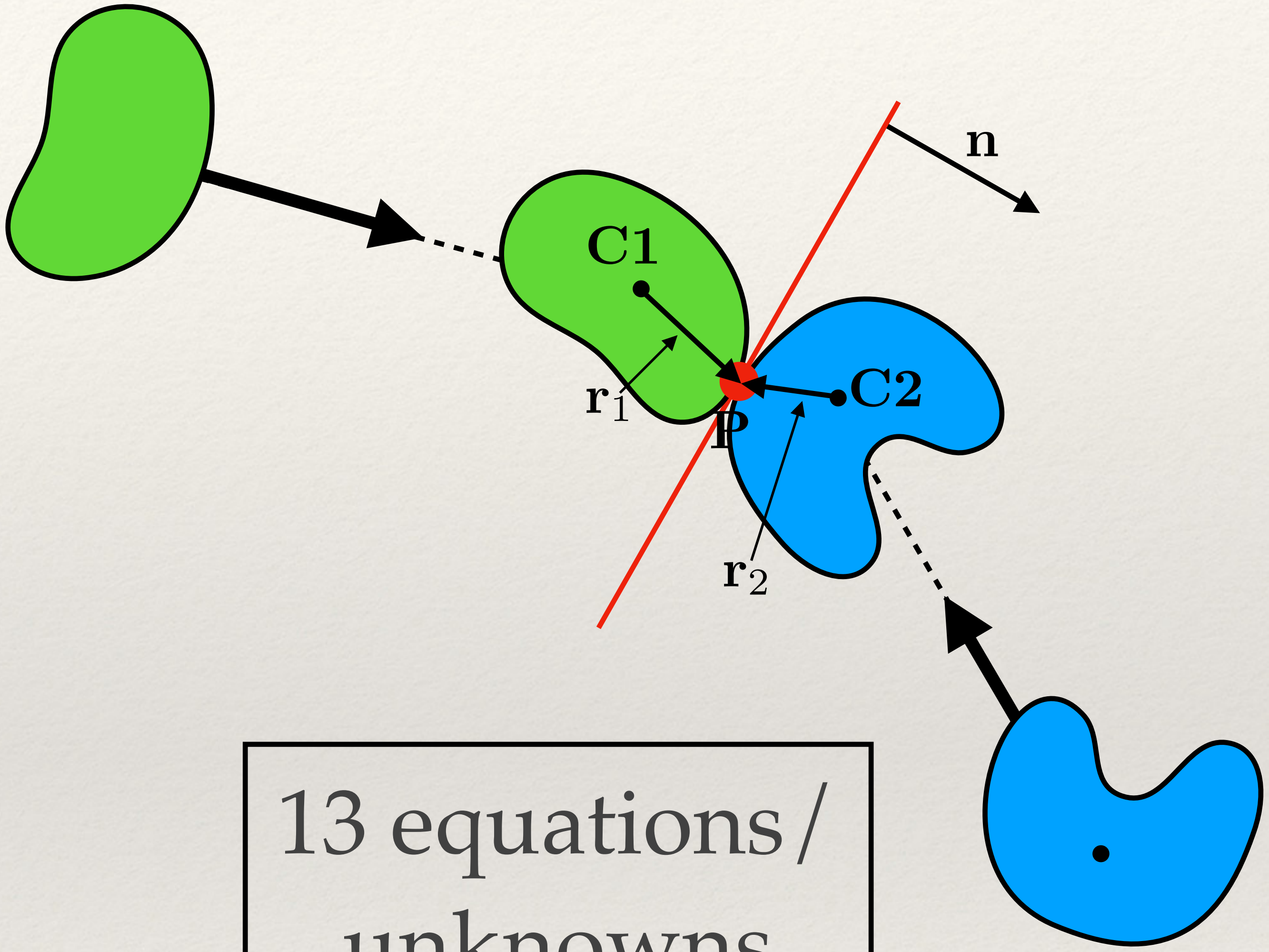
$$m_1 \mathbf{v}'_{C1} = m_1 \mathbf{v}_{C1} + \mathbf{j}$$
$$m_2 \mathbf{v}'_{C2} = m_2 \mathbf{v}_{C2} - \mathbf{j}$$

- Angular momentum

$$I_1 \omega'_1 = I_1 \omega_1 + \mathbf{r}_1 \times \mathbf{j}$$
$$I_2 \omega'_2 = I_2 \omega_2 - \mathbf{r}_2 \times \mathbf{j}$$

- Elastic

$$\mathbf{j} = j\mathbf{n}$$
$$(\mathbf{v}'_{P2} - \mathbf{v}'_{P1}) \cdot \mathbf{n} = -\epsilon(\mathbf{v}_{P2} - \mathbf{v}_{P1}) \cdot \mathbf{n}$$



13 equations / unknowns

Rigid Body Frictional Collision

- Coulomb friction model

μ **coefficient of friction**

Rigid Body Frictional Collision

- Coulomb friction model

  μ **coefficient of friction**

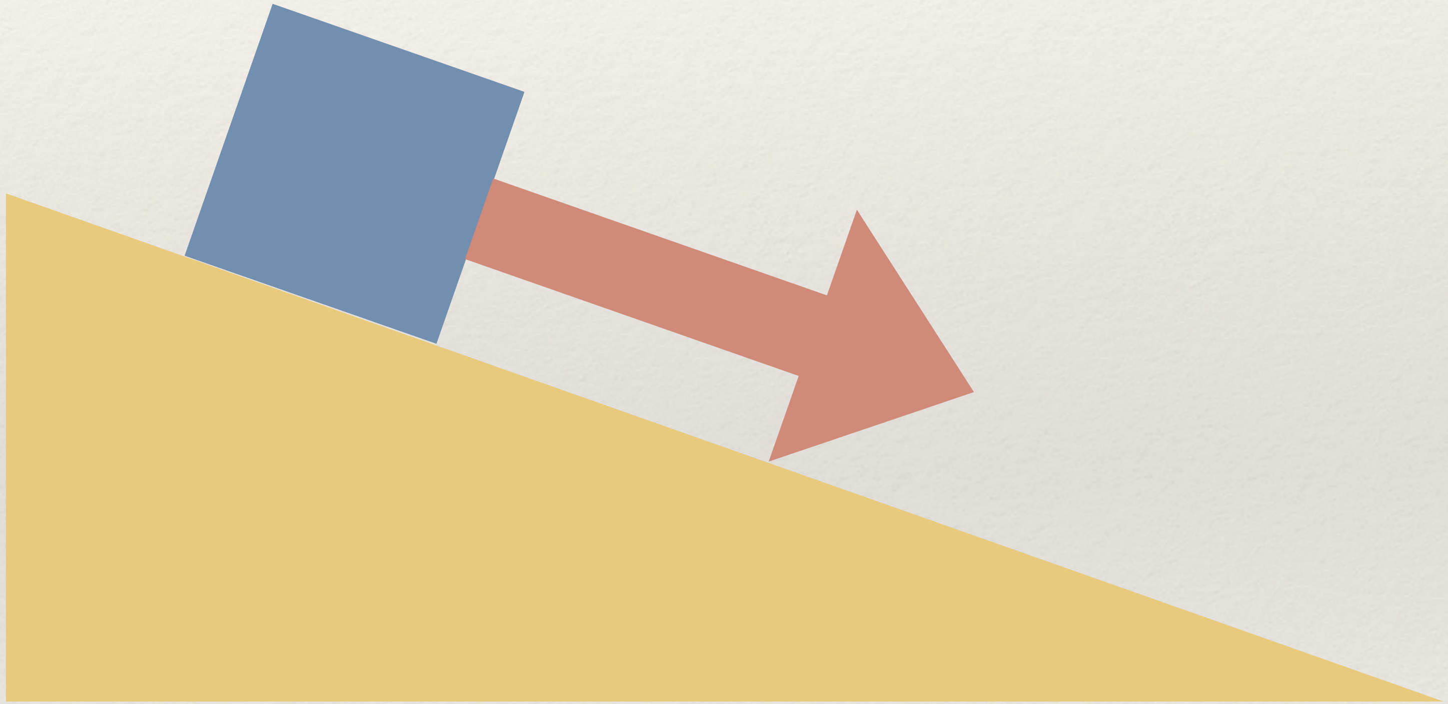
- Static $\|\mathbf{f}_t\| \leq \mu\|\mathbf{f}_n\|$

Rigid Body Frictional Collision

- Coulomb friction model

  μ **coefficient of friction**

- Static $\|\mathbf{f}_t\| \leq \mu \|\mathbf{f}_n\|$
- Sliding $\mathbf{f}_t = -\mu \|\mathbf{f}_n\| \mathbf{t}$



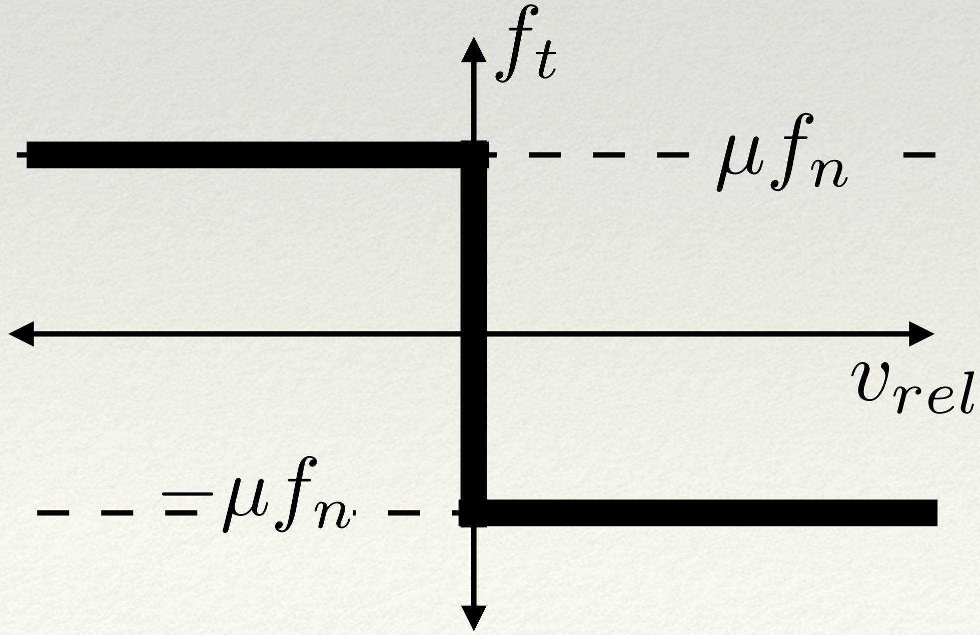
Rigid Body Frictional Collision

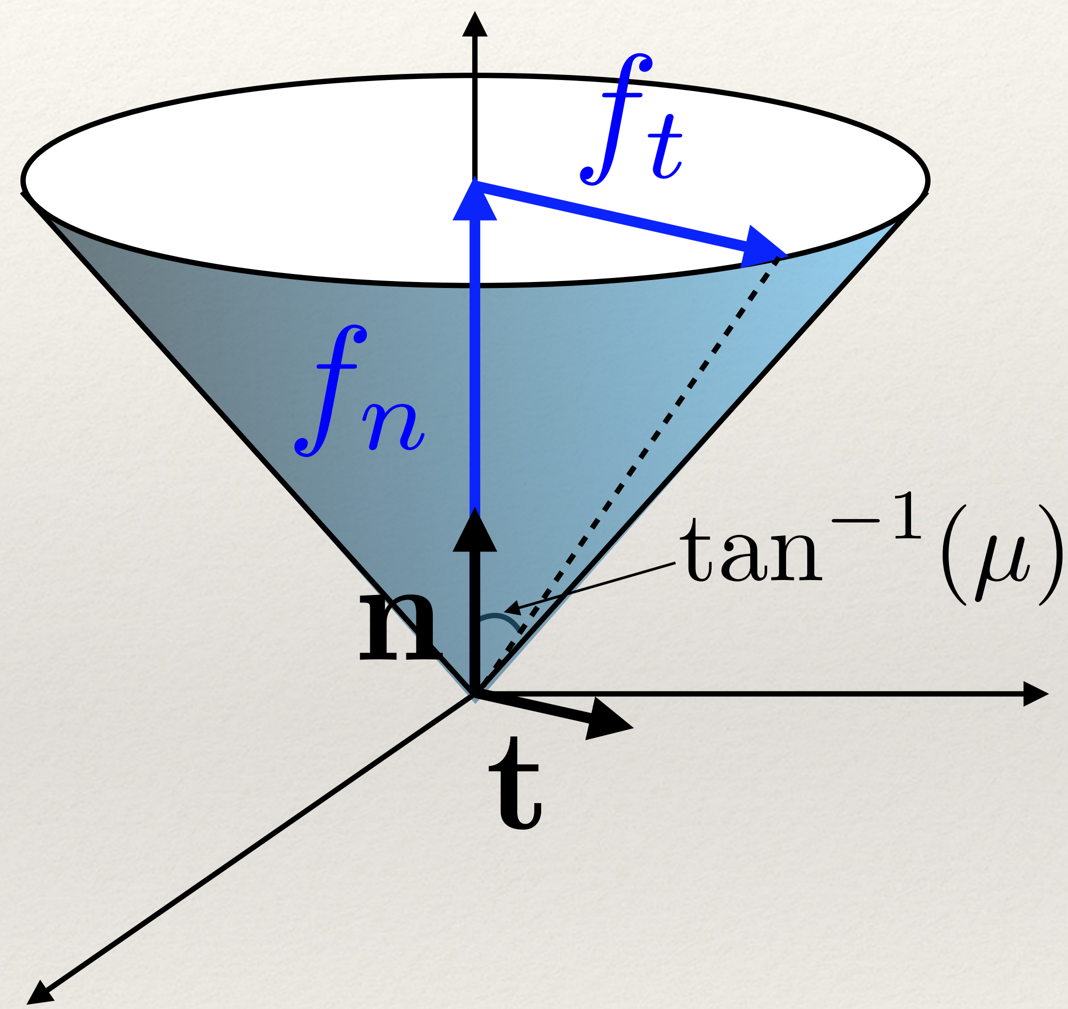
- Coulomb friction model

  μ **coefficient of friction**

- Static $\|\mathbf{f}_t\| \leq \mu\|\mathbf{f}_n\|$

- Sliding $\mathbf{f}_t = -\mu\|\mathbf{f}_n\|\mathbf{t}$

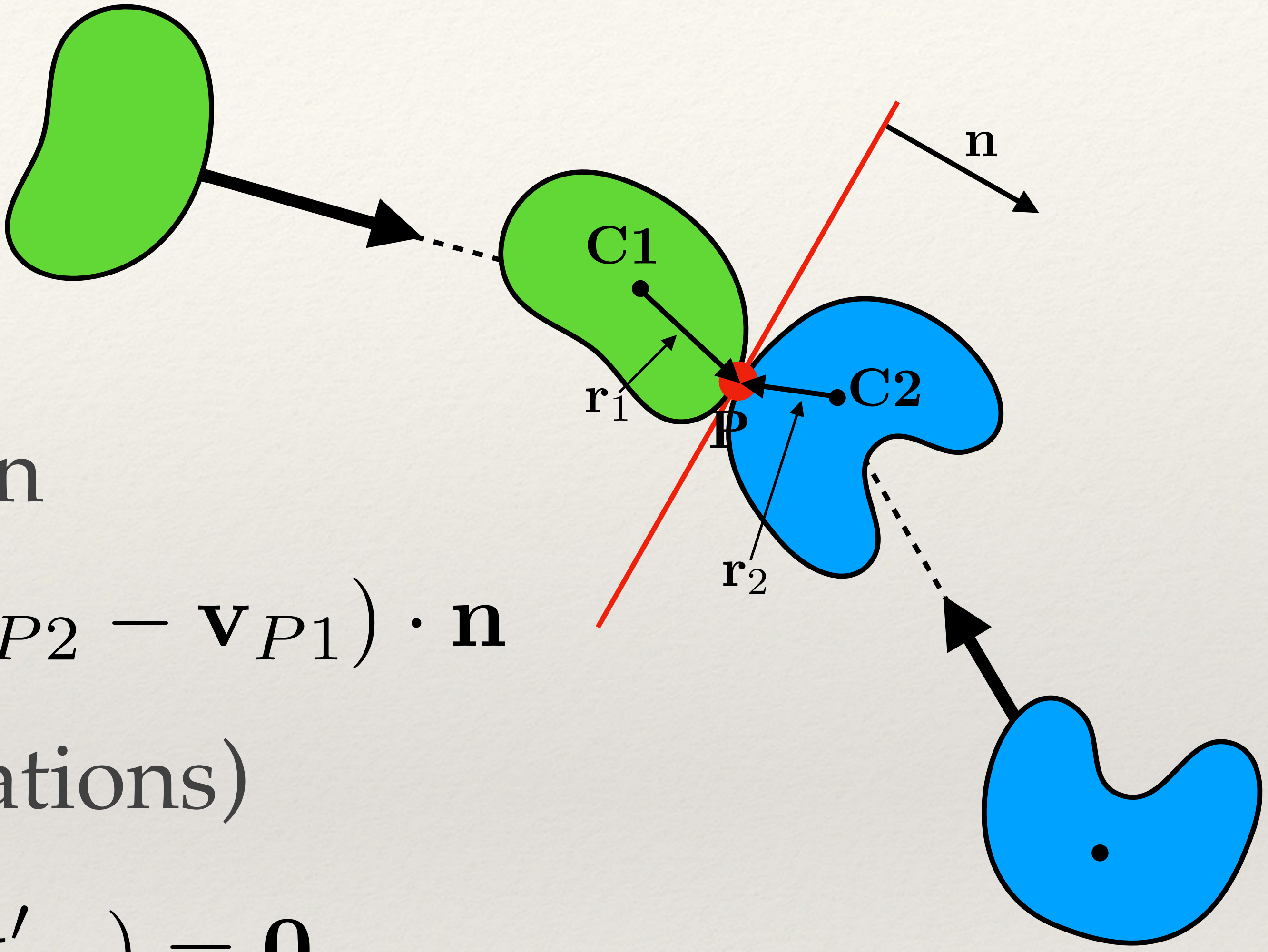




friction cone

Rigid Body Frictional Collision

- Example [Guendelman et al. 2003]



- Elastic in normal direction

$$(\mathbf{v}'_{P2} - \mathbf{v}'_{P1}) \cdot \mathbf{n} = -\epsilon(\mathbf{v}_{P2} - \mathbf{v}_{P1}) \cdot \mathbf{n}$$

- Assume sticking (+2 equations)

$$(I - \mathbf{n}\mathbf{n}^T)(\mathbf{v}'_{P2} - \mathbf{v}'_{P1}) = \mathbf{0}$$

- If not admissible, assume sliding (-2 unknowns)

$$\mathbf{j} = j_n\mathbf{n} - \mu j_n \mathbf{t}$$

Thanks!

For notes, slides, and source code visit:

https://cal.cs.umbc.edu/Courses/PhysicsBasedAnimation

adamb@umbc.edu shinar@cs.ucr.edu