Finite Element Notes

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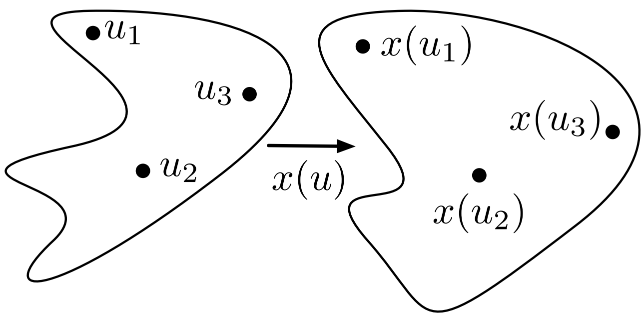


1 Green's Strain

We define a function $\boldsymbol{x}(\boldsymbol{u})$ which describes the mapping from points in a rest configuration or material space to world or deformed space. For each point $\boldsymbol{u}_i$, $\boldsymbol{x}(\boldsymbol{u}_i)$ is the corresponding point after applying the deformation described by $\boldsymbol{x}(\boldsymbol{u})$. If we want to understand how an infinitesimal region around a point is deformed by $\boldsymbol{x}(\boldsymbol{u})$ we look at the deformation gradient $\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{u}}$. The deformation gradient describes how you move in world space as you move through material space. (We will see later why it is called the deformation gradient.)

If our goal is to find elastic forces that undo the deformation, we must first devise a way of measuring deformation. For this we define Green's strain tensor

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial \boldsymbol{x}}{\partial u_i} \cdot \frac{\partial \boldsymbol{x}}{\partial u_j} - \delta_{ij}\right). \tag{1}$$

This metric is non-linear (we'll look at a linear approximation a bit later) and, consequently, is invariant to rotations. Invariance to rotations is *very* good because this means our metric will not respond to rigid body motion of the object.

2 Stress

From Hooke's law, given strain we can define stress, $\boldsymbol{\sigma}$, as

$$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon}, \tag{2}$$

where $\boldsymbol{C}$ is a rank-4 tensor (i.e. a 4-dimensional matrix) with 81 entries that relates the 9 entries in $\boldsymbol{\epsilon}$ to the 9 entries in $\boldsymbol{\sigma}$. Of course, since $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$ are symmetric, these entries are not independent. In fact if we assume that the material is isotropic, there are only two independent parameters and we can write

$$\sigma_{ij} = \lambda \epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij}, \tag{3}$$

where λ and μ are the Lamé constants.

Of course, this is a linear stress-strain relationship. Other relationships are possible (and required when dealing with complex materials).

3 Elastic Potential, Traction, and Force

We can now define the elastic potential energy density, η, as

$$\eta = \frac{1}{2}\sigma_{ij}\epsilon_{ij}. \tag{4}$$

We can also define traction, $\boldsymbol{\tau}$, or force per unit area as

$$\boldsymbol{\tau} = \boldsymbol{\sigma}\boldsymbol{n} \tag{5}$$

where $\boldsymbol{n}$ is the unit surface normal. Force will seek to reduce the energy of the system, thus they will be in the direction of the negative gradient of the energy, that is the force at a point, $\boldsymbol{x}_i$, will be

$$\boldsymbol{f}_i = \frac{\partial \eta}{\partial \boldsymbol{x}_i}. \tag{6}$$

Finite volume methods instead define force by integrating traction over some region,

$$\boldsymbol{f}_i = \oint_{\partial R} \boldsymbol{\sigma}\boldsymbol{n} dS. \tag{7}$$

It has been shown that for the types of finite elements we will be concerned with in this class, these forces are equivalent (and the second form is faster to compute).

4 Damping

Similar functions can be defined for damping, which is dependent on velocity rather than deformation. We define these by taking a time derivative of the above to get

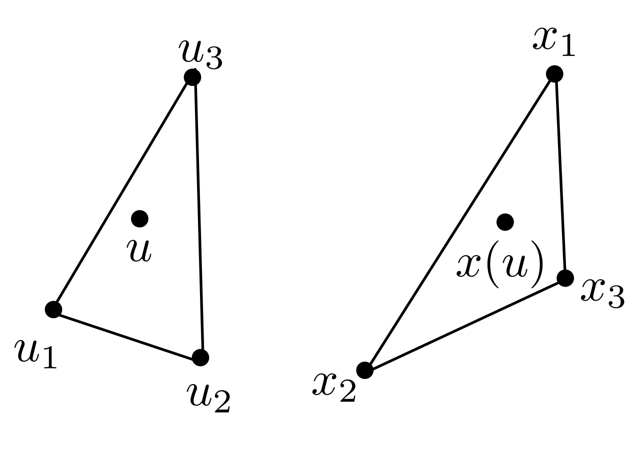
$$v_{ij} = \frac{1}{2}\left(\frac{\partial \boldsymbol{x}}{\partial u_i} \cdot \frac{\partial \dot{\boldsymbol{x}}}{\partial u_j} + \frac{\partial \dot{\boldsymbol{x}}}{\partial u_i} \cdot \frac{\partial \boldsymbol{x}}{\partial u_j}\right) \tag{8}$$

$$\sigma^v_{ij} = \phi v_{kk}\delta_{ij} + 2\psi v_{ij} \tag{9}$$

$$\kappa = \frac{1}{2}\sigma^v_{ij} v_{ij} \tag{10}$$

5 Linear Finite Elements

As the name implies finite element methods take an object we wish to simulate and break it up into a finite set of pieces. While arbitrary elements are possible, we'll stick to simplicies (triangles in 2D tetrahedra in 3D).



Finite elements work by essentially limiting the types of functions that can be represented. To do this we define a basis over each element. We can then work with functions expressed in this basis. The obvious basis to use with simplicies is the linear basis we should all be familiar with: barycentric coordinates. Recall, that a point, u in a triangle can be expressed as a convex combination of the vertices of the triangle,

$$\boldsymbol{u} = b_1\boldsymbol{u}_1 + b_2\boldsymbol{u}_2 + b_3\boldsymbol{u}_3 \tag{11}$$

However, this is redundant since we have the additional constraint that $b_1 + b_2 + b_3 = 1$. So, we can alternately write,

$$\boldsymbol{u} = b_1\boldsymbol{u}_1 + b_2\boldsymbol{u}_2 + (1-b_1-b_2)\boldsymbol{u}_3 = \boldsymbol{u}_3 + b_1(\boldsymbol{u}_1-\boldsymbol{u}_3) + b_2(\boldsymbol{u}_2-\boldsymbol{u}_3), \tag{12}$$

or in matrix form

$$\boldsymbol{u} = \boldsymbol{u}_3 + \begin{pmatrix} (u_1-u_3)_x & (u_2-u_3)_x \\ (u_1-u_3)_y & (u_2-u_3)_y \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{13}$$

Thus, we construct a matrix, which we call B, that contains vectors along the edges of the triangle as its columns and this matrix describes the mapping from barycentric coordinates to material coordinates. If we wish to go the other way, we will need to invert this matrix.

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \boldsymbol{B}^{-1}(\boldsymbol{u}-\boldsymbol{u}_3) = \boldsymbol{\beta}(\boldsymbol{u}-\boldsymbol{u}_3) \tag{14}$$

Letting $\boldsymbol{\beta} = \boldsymbol{B}^{-1}$. Similarly, when mapping from barycentric coordinates to world coordinates we have

$$\boldsymbol{x} = \boldsymbol{x}_3 + \boldsymbol{X}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{15}$$

where $\boldsymbol{X}$ is a matrix made up of vectors along the edges of the triangle in world space. Now we can define the entire mapping as

$$\boldsymbol{x}(\boldsymbol{u}) = \boldsymbol{x}_3 + \boldsymbol{X}\beta(\boldsymbol{u}-\boldsymbol{u}_3) \tag{16}$$

Taking the gradient (derivative with respect to u) of this function we have

$$\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{u}} = \boldsymbol{X}\boldsymbol{\beta}. \tag{17}$$

This is the deformation gradient. For linear finite elements, it is a matrix which we call $\boldsymbol{F}$. We can similarly define the time derivative as

$$\frac{\partial \dot{\boldsymbol{x}}}{\partial \boldsymbol{u}} = \boldsymbol{V}\boldsymbol{\beta} \tag{18}$$

Where $\boldsymbol{V}$ contains velocity differences rather than position differences in world coordinates. Finally, we can write Green's strain as

$$\boldsymbol{\epsilon} = \frac{1}{2}\left(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\right) \tag{19}$$

6 Cauchy's Infinitesimal Strain

Suppose we write our deformation gradient as

$$\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{u}} = \boldsymbol{I} + \boldsymbol{D} \tag{20}$$

That is the deformation gradient is the identity (no deformation) plus some amount of deformation. Taking Green's strain we have

$$\begin{aligned} \boldsymbol{\epsilon} &= \frac{1}{2}((\boldsymbol{I}+\boldsymbol{D})^T(\boldsymbol{I}+\boldsymbol{D}) - \boldsymbol{I}) && (21) \\ &= \frac{1}{2}(\boldsymbol{I}\boldsymbol{I} + \boldsymbol{D}^T + \boldsymbol{D} + \boldsymbol{D}^T\boldsymbol{D} - \boldsymbol{I}) && (22) \\ &= \frac{1}{2}(\boldsymbol{D}^T + \boldsymbol{D}). && (23) \end{aligned}$$

Now, if $\boldsymbol{D}$ is very small then $\boldsymbol{D}^T\boldsymbol{D}$ is much smaller than $\boldsymbol{D}$ and $1/2(\boldsymbol{D}+\boldsymbol{D}^T)$ is a good estimate of Green's strain. Furthermore, this strain measure is linear which leads to all sorts of nice consequences. Unfortunately, it is not invariant to rotations, which leads to artifacts if it is used for large deformations. Cauchy's strain can be written as

$$\boldsymbol{\epsilon} = \frac{1}{2}\left(\frac{\partial x_i}{\partial u_j} + \frac{\partial x_j}{\partial u_i}\right) - \delta_{ij} = \frac{1}{2}\left(\boldsymbol{F} + \boldsymbol{F}^T\right) - \boldsymbol{I} \tag{24}$$

7 Other Stress Measures

We can see from the definition of traction

$$\boldsymbol{\tau} = \boldsymbol{\sigma}\boldsymbol{n} \tag{25}$$

that stress maps normals to forces. However, its important to distinguish where these normals and forces are defined. If both normals and forces are in world space, the stress is know as a *Cauchy* stress and often written as $\boldsymbol{\sigma}$. If the stress maps normals in material space to forces in material space it is known as a Second Piola-Kirchhoff stress and sometimes written as $\boldsymbol{S}$. A first Piola-Kirchhoff stress maps normals in material space to forces in world space (as such it is especially convenient) and is written $\boldsymbol{P}$. Now, watch out, when only one stress is being considered it is often written with $\boldsymbol{\sigma}$. The stress we defined earlier was actually a second Piola-Kirchhoff stress. Now its easy to convert between these stress since they all measure the same thing just in different coordinate systems. Letting $J = \det(\boldsymbol{F})$ we have

$$\boldsymbol{P} = J\sigma\boldsymbol{F}^{-T}, \tag{26}$$

and

$$\boldsymbol{P} = \boldsymbol{F}\boldsymbol{S}. \tag{27}$$

By having different ways of specifying stress, we can choose whichever one is most convenient for a given application. The first Piola-Kirchhoff is particularly attractive since it works with normals in the material space (where they are constant) and maps directly to forces in world space (where they will be applied).

8 Computing Forces

Using Einstein's summation convention we can write

$$\frac{\partial x_i}{\partial u_j} = X_{ik}\beta_{kj} \tag{28}$$

$$\begin{aligned} \epsilon_{ij} &= \frac{1}{2}\left(\frac{\partial x_a}{\partial u_i} \cdot \frac{\partial x_a}{\partial u_j} - \delta_{ij}\right) && (29) \\ &= \frac{1}{2}X_{mk}\beta_{ki}X_{mn}\beta_{nj} + \frac{1}{2}\delta_{ij} && (30) \end{aligned}$$

$$
\begin{aligned}
\sigma_{ij} &= \lambda \epsilon_{pp}\delta_{ij} + 2\mu\epsilon_{ij} && (31)\\
&= \frac{\lambda}{2}\left(X_{mk}\beta_{kp}X_{mn}\beta_{np} - 3\right) && (32)\\
&\quad +\mu\left(X_{mk}\beta_{ki}X_{mn}\beta_{nj} - \delta_{ij}\right) && (33)
\end{aligned}
$$

$$
\begin{aligned}
\eta &= \frac{1}{2}\epsilon_{ij}\sigma_{ij} && (34)\\
&= \frac{1}{2}\left(\frac{1}{2}X_{mk}\beta_{ki}X_{mn}\beta_{nj} + \frac{1}{2}\delta_{ij}\right) && (35)\\
&\quad \left(\frac{\lambda}{2}\left(X_{mk}\beta_{kp}X_{mn}\beta_{np} - 3\right)\right. && (36)\\
&\quad \left. + \mu\left(X_{mk}\beta_{ki}X_{mn}\beta_{nj} - \delta_{ij}\right)\right) && (37)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \eta}{\partial X_{ba}} &= \frac{1}{2}\left(\frac{1}{2}\delta_{bm}\delta_{ak}\beta_{ki}X_{mn}\beta_{nj}\right. && (38)\\
&\quad \left.+\frac{1}{2}X_{mk}\beta_{ki}\delta_{bm}\delta_{an}\beta_{nj}\right)\sigma_{ij} && (39)\\
&\quad +\frac{1}{2}\epsilon_{ij}\left(\frac{\lambda}{2}\delta_{bm}\delta_{ak}\beta_{kp}X_{mn}\beta_{np}\delta_{ij}\right. && (40)\\
&\quad +\frac{\lambda}{2}X_{mk}\beta_{kp}\delta_{bm}\delta_{an}\beta_{np}\delta_{ij} && (41)\\
&\quad +\mu\delta_{bm}\delta_{ak}\beta_{ki}X_{mn}\beta_{nj} && (42)\\
&\quad \left.+\mu X_{mk}\beta_{ki}\delta_{mb}\delta_{an}\beta_{nj}\right) && (43)\\
&= \frac{1}{4}\left(\beta_{ai}X_{bn}\beta_{nj} + X_{bk}\beta_{ki}\beta_{aj}\right)\sigma_{ij} && (44)\\
&\quad +\frac{1}{2}\epsilon_{ij}\left(\frac{\lambda}{2}\beta_{ap}X_{bn}\beta_{np}\delta_{ij}\right. && (45)\\
&\quad +\frac{\lambda}{2}X_{bk}\beta_{kp}\beta_{ap}\delta_{ij} && (46)\\
&\quad \left.+\mu\beta_{ai}X_{bn}\beta_{nj} + \mu X_{bk}\beta_{ki}\beta_{aj}\right) && (47)\\
&= \frac{1}{2}X_{nb}\beta_{ai}\beta_{nj}\sigma_{ij} && (48)\\
&\quad +\frac{1}{2}\epsilon_{ij}\left(\lambda X_{bn}\beta_{np}\beta_{ap}\delta_{ij}\right. && (49)\\
&\quad \left.+2\mu X_{bn}\beta_{ai}\beta_{n}j\right) && (50)\\
&= \frac{1}{2}X_{nb}\beta_{ai}\beta_{nj}\sigma_{ij} && (51)\\
&\quad +\frac{1}{2}\left(2X_{nb}\beta_{ai}\beta_{nj}\right) && (52)\\
&\quad \left(\frac{\lambda}{2}\epsilon_{kk}\delta_{ij} + \mu\epsilon_{ij}\right) && (53)\\
&= X_{nb}\beta_{ai}\beta_{nj}\left(\frac{1}{2}\sigma_{ij} + \frac{1}{2}\sigma_{ij}\right) && (54)\\
&= X_{nb}\beta_{ai}\beta_{nj}\sigma_{ij} && (55)\\
& && (56)
\end{aligned}
$$

Reintroducing summations and integrating over the volume of the element (in material space), v, we have the force on node a is

$$
\boldsymbol{f}_a = -v\sum_{n=1}^{3}\boldsymbol{X}_{[n]}\sum_{i=1}^{3}\sum_{j=1}^{3}\beta_{nj}\beta_{ai}\sigma_{ij} \qquad (57)
$$

An alternate formulation (from finite volumes) yields the following forces

$$
\boldsymbol{f}_a = -\frac{1}{3}\boldsymbol{F}\boldsymbol{\sigma}(a_1\boldsymbol{n}_1 + a_2\boldsymbol{n}_2 + a_3\boldsymbol{n}_3), \qquad (58)
$$

where $a_i\boldsymbol{n}_i$ are the area-weighted normals of the three faces incident to node a.

9 A Few Other Useful Formulas

We need the gradient of force for the stiffness matrix ($\boldsymbol{K}$). A derivation similar to the above yields:

$$
\begin{aligned}
\frac{\partial F_{ai}}{\partial X_{pe}} &= \delta_{pi}\beta_{ej}\beta_{ak}\sigma_{kj} && (59)\\
&+ \lambda X_{ib}\beta_{bk}\beta_{ak}\beta_{em}X_{pd}\beta_{dm} && (60)\\
&+ \mu X_{ib}\beta_{bj}\beta_{ak}\beta_{ej}X_{pd}\beta_{dk} && (61)\\
&+ \mu X_{ib}\beta_{bj}\beta_{ak}X_{pc}\beta_{cj}\beta_{ek} && (62)
\end{aligned}
$$

The change in force with respect to velocity is

$$
\begin{aligned}
\frac{\partial F_{ai}}{\partial \dot{X}_{pe}} &= \phi X_{ib}\beta_{bk}\beta_{ak}\beta_{em}X_{pd}\beta_{dm} && (63)\\
&+ \psi X_{ib}\beta_{bj}\beta_{ak}\beta_{ej}X_{pd}\beta_{dk} && (64)\\
&+ \psi X_{ib}\beta_{bj}\beta_{ak}X_{pc}\beta_{cj}\beta_{ek} && (65)
\end{aligned}
$$

For linear strain we have $\boldsymbol{X\beta} = \boldsymbol{I}$ and zero stress the formulas simplify to

$$
\begin{aligned}
\frac{\partial F_{ai}}{\partial X_{pe}} &= \lambda\delta_{ik}\beta_{ak}\beta_{em}\delta_{pm} && (66)\\
&+ \mu\delta_{ij}\beta_{ak}\beta_{ej}\delta_{pd} && (67)\\
&+ \mu\delta_{ij}\beta_{ak}\delta_{pj}\beta_{ek} && (68)
\end{aligned}
$$

The change in force with respect to velocity is

$$
\begin{aligned}
\frac{\partial F_{ai}}{\partial \dot{X}_{pe}} &= +\phi\delta_{ik}\beta_{ak}\beta_{em}\delta_{pm} && (69)\\
&+ \psi\delta_{ij}\beta_{ak}\beta_{ej}\delta_{pd} && (70)\\
&+ \psi\delta_{ij}\beta_{ak}\delta_{pj}\beta_{ek} && (71)
\end{aligned}
$$